Viscoacoustic modeling and imaging using low-rank approximation

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ABSTRACT

A constant-Q wave equation involving fractional Laplacians was recently introduced for viscoacoustic modeling and imaging. This fractional wave equation has a convenient mixed-domain space-wavenumber formulation, which involves the fractional-Laplacian operators with a spatially varying power. We have applied the low-rank approximation to the mixed-domain symbol, which enables a space-variable attenuation specified by the variable fractional power of the Laplacians. Using the proposed approximation scheme, we formulated the framework of the Q-compensated reverse time migration (Q-RTM) for attenuation compensation. Numerical examples using synthetic data demonstrated the improved accuracy of using low-rank wave extrapolation with a constant-Q fractional-Laplacian wave equation for seismic modeling and migration in attenuating media. Low-rank Q-RTM applied to viscoacoustic data is capable of producing images comparable in quality with those produced by conventional RTM from acoustic data.

INTRODUCTION

Modeling seismic wave propagation in attenuating media accounts for the effective anelastic characteristics of the real earth (Carcione, 2007). Numerous studies have shown that many of the hydrocarbon prospecting areas, such as those where gas accumulations are present, strongly attenuate seismic waves (Dvorkin and Mavko, 2006). Seismic attenuation can be expressed as a combined effect of energy loss and velocity dispersion.

Attenuation effects can be modeled by incorporating the quality factor Q in the time-domain wave equation. One of the classic approaches involves a superposition of mechanical elements (e.g., Maxwell and standard linear solid [SLS] elements) to characterize Q, and it is known as the approximate constant-Q model (Liu et al., 1976; Blanch et al., 1995; Carcione, 2007; Zhu et al., 2013). The approximate constant-Q approach suffers from large computational and memory requirements. Kjartansson (1979) initially proposes a constant-Q model that assumes a linear relationship between the attenuation coefficient and frequency. This model was proven accurate in capturing a constant Q behavior within the seismic frequency band. However, early implementations of the constant-Q model involved a fractional time derivative, which required storing the whole history of the wavefield (Caputo and Mainardi, 1971). This requirement rendered the memory cost too high for practical applications, even when the fractional operator was truncated after a certain time period (Podlubny, 1999; Carcione et al., 2002; Carcione, 2009). To overcome this issue, fractional-Laplacian operators (Chen and Holm, 2004) have been introduced to approximate the constant-Q viscoacoustic wave equation (Carcione, 2010; Zhu and Harris, 2014). The fractional-Laplacian approach is attractive because it can be conveniently formulated in the wavenumber domain using Fourier transforms and without introducing any extra equations or variables (Carcione, 2010). Using this approach, Zhu and Harris (2014) develop a decoupled wave equation that accounts separately for amplitude attenuation and phase dispersion effects, thus allowing for correct compensation for both factors during back propagation by reversing the sign of the attenuation operator and keeping the sign of the dispersion operator unchanged (Zhu, 2014).

which incorporates a single relaxation mechanism (Robertsson et al., 1994; Blanch et al., 1995).

The fractional-Laplacian approach was previously implemented using either a pseudospectral method, by averaging the fractional power of the Laplacian operator as an approximation (Zhu and Harris, 2014), or a finite-difference approach (Lin et al., 2009). We apply a low-rank approximation scheme (Fomel et al., 2013; Sun and Fomel, 2013) to implement decoupled fractional Laplacians of Zhu and Harris (2014) in wave extrapolation with the goal of accurately capturing spatially varying fractional power. The advantage of the low-rank approach is its ability to directly approximate the mixed-domain wave extrapolation operator with a separable representation, which minimizes the number of fast Fourier transforms (FFTs) per time step. In addition, we derive the adjoint of the forward modeling operator, which correctly compensates for velocity dispersion but not amplitude loss. The proposed adjoint of the forward modeling operator, which correctly compensates for amplitude loss during back propagation of the viscoacoustic data (Zhu et al., 2014). When using an operator that compensates for amplitude loss during back propagation and can be supplied to the low-rank one-step wave extrapolation algorithm (Sun and Fomel, 2013).

To compensate for attenuation, reversing the sign of the second term on the right side of equation 5 can amplify the amplitude, whereas the other term must be kept unchanged to counteract the dispersion effects (Zhu, 2014). Thus, the attenuation-compensated constant-Q wave equation corresponds to the dispersion relation:

$$\frac{c^2}{\varepsilon^2} = -\eta|k|^{2r+2} + i\alpha r |k|^{2r+1},$$

which corresponds to the following constant-Q wave equation with fractional Laplacians:

$$\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \nabla^2 P + \{\eta(-\nabla^2)^{r+1} - \nabla^2\}P + \tau \frac{\partial (-\nabla^2)^{r+1/2}P}{\partial t},$$

where $\eta = -c_0^2\omega_0^{-2r} \cos(\pi r)$ and $\tau = c_0^{-2r} a_0^{-2r} \sin(\pi r)$. Note that $c_0$ (the phase velocity) and $\gamma$ (the fractional power) can be heterogeneous (depending on $x$). Solving for $\omega$ in equation 5 yields

$$\omega = \frac{-i p_1 + p_2}{2},$$

where

$$p_1 = \tau c^2 |k|^{2r+1},$$

$$p_2 = \sqrt{-\tau^2 c^4 |k|^{4r+2} - 4\eta c^2 |k|^{2r+2}}.$$  

The phase function $\phi(x, k, \Delta t)$ that determines the phase shift of wavefield for propagation in time is then defined as

$$\phi_1(x, k, \Delta t) = k \cdot x + \frac{-i p_1 + p_2}{2} \Delta t$$

and can be supplied to the low-rank one-step wave extrapolation algorithm (Sun and Fomel, 2013).

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$$\frac{c^2}{\varepsilon^2} = -\eta|k|^{2r+2} + i\alpha r |k|^{2r+1},$$

which defines the complex-conjugate phase function:

$$\phi_2(x, k, \Delta t) = \phi_1(x, k, \Delta t) = k \cdot x + \frac{-i p_1 + p_2}{2} \Delta t.$$  

The $\phi_1$ and $\phi_2$ involve the fractional power of wavenumber, and depend on $x$ and $k$, the Fourier transform pair. The low-rank one-step wave extrapolation operator (Sun and Fomel, 2013) provides a convenient way to use the phase function to extrapolate a viscoacoustic wavefield, while allowing $\gamma(x)$, the fractional power coefficient, to vary in space. The one-step mixed-domain operator has the form of the following Fourier integral operator:

$$P(x, t + \Delta t) = \int \hat{P}(k, t) e^{i \phi(x, k, \Delta t)} \, dk,$$

where $\hat{P}$ is the spatial Fourier transform of $P$. Its adjoint form can be expressed as

THEORY

A constant-Q model assumes that the attenuation coefficient is linear in frequency (Kjartansson, 1979). The time-domain viscoacoustic wave equation for constant-Q model can be written as (Carcione et al., 2002)

$$\frac{\partial^2 P}{\partial t^2} = c^2 \omega_0^{-2r} \nabla^2 P,$$

where $\nabla^2$ is the Laplacian operator and $P(x, t)$ is the pressure wavefield, and

$$\gamma(x) = \arctan(1/Q(x))/\pi$$

is a dimensionless parameter:

$$c^2(x) = c_0^2(x) \cos^2(\pi\gamma(x)/2),$$

where $c_0(x)$ is the velocity model defined at the reference frequency $\omega_0$ and $x$ is the spatial coordinate. When $Q$ is finite, $\gamma$ is greater than zero and the wave equation involves a fractional time derivative.

In a homogeneous medium, considering the plane-wave solution $e^{i(\omega t - k \cdot x)}$, where $\omega$ is the angular frequency, and $k$ is the complex wavenumber vector and substituting it into equation 1, leads to the dispersion relation as follows:

$$\frac{c^2}{\varepsilon^2} = (i)^{2r} \omega_0^{-2r} \omega^{2r} |k|^2.$$

Starting from equation 4, Zhu and Harris (2014) derive the following approximate dispersion relation:

$$\frac{\omega^2}{c^2} = -\eta|k|^{2r+2} - i\alpha r |k|^{2r+1}. $$

A constant-Q wave equation with fractional Laplacians:

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where $\eta = -c_0^2\omega_0^{-2r} \cos(\pi r)$ and $\tau = c_0^{-2r} a_0^{-2r} \sin(\pi r)$. Note that $c_0$ (the phase velocity) and $\gamma$ (the fractional power) can be heterogeneous (depending on $x$). Solving for $\omega$ in equation 5 yields

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$$P(x, t + \Delta t) = \int \hat{P}(k, t) e^{i \phi(x, k, \Delta t)} \, dk,$$

where $\hat{P}$ is the spatial Fourier transform of $P$. Its adjoint form can be expressed as

$$\frac{\omega^2}{c^2} = -\eta|k|^{2r+2} + i\alpha r |k|^{2r+1}.$$
\[ \hat{P}(k,t) = \int P(x,t) e^{-i\tilde{\phi}(x,k) \Delta t} dk, \] (14)

where \( \tilde{\phi} \) denotes the complex conjugate of \( \phi \). Substituting equation 10 into equation 13, we can write the forward extrapolation operator explicitly as

\[ P(x,t+\Delta t) = \int \hat{P}(k,t) e^{i\phi(x,k) \Delta t} dk \]

\[ = \int \hat{P}(k,t) e^{ikx + (p_1 + ip_2) \Delta t/2} dk. \] (15)

The corresponding adjoint operator takes the form

\[ \hat{P}_{\text{adj}}(k,t) = \int P(x,t+\Delta t) e^{-i\hat{\phi}(x,k) \Delta t} dk \]

\[ = \int P(x,t+\Delta t) e^{-ikx + (p_1 - ip_2) \Delta t/2} dk. \] (16)

To make the computation feasible, we apply low-rank decomposition proposed by Fomel et al. (2013) to approximate the wave extrapolation symbol in equations 15 and 16.

Note that the adjoint operator compensates for velocity dispersion. Because the sign of \( p_1 \) remains unchanged, the amplitude of waves will be attenuated during backward propagation using equation 16. This means that the adjoint operator may not be well suited for RTM because the backward wavefield would be attenuated twice. However, the adjoint operator can be supplied into the framework of least-squares RTM, which can then iteratively recover the amplitude loss because of viscoacoustic attenuation (Sun et al., 2014).

An alternative strategy for backward wave propagation is to try compensating for the full attenuation effect (amplitude loss and velocity dispersion) using the operator

\[ P_{\text{comp}}(x,t+\Delta t) = \int \hat{P}(k,t) e^{i\phi_2(x,k) \Delta t} dk \]

\[ = \int \hat{P}(k,t) e^{ikx - (p_1 + ip_2) \Delta t/2} dk. \] (17)

which is the adjoint of

\[ \hat{P}_{\text{comp}}(k,t) = \int P(x,t+\Delta t) e^{-i\phi_2(x,k) \Delta t} dk \]

\[ = \int P(x,t+\Delta t) e^{-ikx - (p_1 - ip_2) \Delta t/2} dk. \] (18)

In application to RTM, the operator in equation 17 has an exponentially growing term that can amplify the energy in the forward-propagated source wavefield. In the same manner, the operator in equation 18 can compensate for the amplitude loss in the backward-propagated receiver wavefield. To avoid magnifying high-frequency components during wave propagation, we use a low-pass Tukey filter for the attenuation and dispersion operators in the wavenumber domain (Zhu et al., 2014). A complex-valued imaging condition that compensates for velocity dispersion and amplitude loss

![Figure 1](https://library.seg.org/)
can be obtained by crosscorrelating the source and receiver wavefields modeled using equations 17 and 18 (Zhu et al., 2014).

In this way, the migrated viscoacoustic data get corrected for attenuation because of the viscoacoustic material encountered along the wavepath. As will be demonstrated in the numerical examples, $Q$-compensated RTM is capable of enhancing illumination in areas where attenuating material is present. For a more accurate compensation of attenuation, operators in equations 17 and 18 can be used to design a preconditioner in iterative least-squares RTM.

**NUMERICAL EXAMPLES**

Two-layer model

The purpose of our first example is to investigate the accuracy of the solution of the constant-$Q$ wave equation using the proposed low-rank scheme in the presence of a sharp contrast in velocity and $Q$. We use an isotropic two-layer model with $v = 1800$ m/s in the top layer and $v = 3600$ m/s in the bottom layer. The model is discretized on a 200 × 200 grid, with a spatial sampling of 8 m along the x- and z-directions. An explosive source with a peak frequency of 50 Hz is located at the center. The reference frequency is $\omega_0 = 1500$ Hz. Wavefield snapshots are taken at $t = 330$ ms. Figure 1a shows the acoustic case, in which the model has the velocity discontinuity but no attenuation ($Q = \infty$). For comparison, Figure 1b demonstrates the effect of homogeneous attenuation, where $Q = 30$. Both velocity dispersion and amplitude loss can be observed. In Figure 1c, we set $Q = 30$ in the top layer and $Q = 100$ in the bottom layer. The transmitted arrival exhibits less attenuation compared with that in Figure 1b. In Figure 1d, velocity and $Q$ remain the same as those in Figure 1b; however, the fractional power of Laplacians $\gamma$ is taken to be the averaged value, which corresponds to the original implementation by Zhu and Harris (2014). To compare the results modeled by the two strategies, a middle trace at $x = 800$ m is extracted from both wavefield snapshots (Figure 1c and 1d). Figure 2 shows the two traces, along with their difference. Errors caused by using a constant $\gamma$ instead of a spatially varying $\gamma$ can be easily observed.

Marmousi $Q$-model

In the second example, we use a synthetic attenuation model to investigate the effect of $Q$ compensation in RTM. Figure 3a shows the Marmousi velocity model. We build a corresponding $Q$ model (Figure 3b) on the same grid. The model features three highly attenuative zones in the shallow parts of the model. This kind of an attenuation pattern can be caused in reality by the presence of a gas accumulation. The model is discretized on a 241 × 961 grid with a spacing of 12.5 m in horizontal and vertical directions. A total of 64 shots with a horizontal spacing of 187.5 m was used, starting from 25 m, and the source is a Ricker wavelet with a peak frequency of 20 Hz. Receivers have a spacing of 12.5 m, starting from 0 m and ending at 12,000 m. For simplicity of modeling, sources and receivers are located at −62.5 m depth. The data have a temporal sampling of 2 ms with a total length of 8 s. First, acoustic modeling
is used to generate acoustic data (Figure 4a), and then viscoacoustic modeling is used to generate viscoacoustic data (Figure 4b), accounting for seismic attenuation during wave propagation. We first apply acoustic RTM on the (nonattenuated) acoustic data to generate a reference image. The image generated by noncompensated RTM using viscoacoustic data (Figure 5a) suffers from a lack of illumination within and below the attenuative region. Using the proposed $Q$-RTM according to equations 17 and 18 (Figure 5b), the amplitude of the parallel normal faults and anticline structures has been recovered to a large extent, and image resolution inside and below gas has been greatly enhanced. The Tukey filter is applied with a cutoff frequency of 100 Hz with a taper ratio of 0.4. We extract the trace at $x = 6875$ m from the three images and use the acoustic RTM image from migrating acoustic data as a reference.

Figure 4. Common-shot gathers used for RTM and $Q$-RTM at $x = 6025$ m. A total length of 8 s has been recorded, whereas only the first 5 s is displayed. (a) Acoustic data, (b) viscoacoustic data, and (c) viscoacoustic data with added random noise ($S/N = 0.9$). The right panel shows a trace extracted from $x = 4375$ m.

Figure 5. A portion of RTM images obtained using different approaches. (a) Acoustic RTM image using viscoacoustic data, (b) $Q$-RTM image using viscoacoustic data, and (c) $Q$-RTM image using viscoacoustic data with added noise. The right panel compares the trace at $x = 6875$ m from the corresponding image with the reference image obtained by applying acoustic RTM on acoustic data. The blue solid line refers to the reference image, the black solid line refers to the noncompensated image, the red dashed line refers to the compensated image, and the pink dashed line refers to the compensated image using noisy data.
also shown in Figure 5a and 5b). The comparison shows that the conventional RTM applied to viscoacoustic data suffers from the effects of phase dispersion and amplitude attenuation, especially in the deeper part of the image, whereas the Q-compensated image highly resembles the nonattenuated acoustic image in amplitude and phase. To test the sensitivity of the proposed method to noise in the data, we add Gaussian random noise to the viscoacoustic data with a signal-to-noise ratio (S/N) of 0.9 (Figure 4c). Applying Q-RTM to the noisy data leads to a slightly noisier image (Figure 5c). However, migration remains stable and all the reflectors remain well imaged.

**DISCUSSION**

For acoustic RTM and Q-RTM in the Marmousi example with a time step size of 2 ms, we used the low-rank approximation with a rank of four, corresponding to five complex-to-complex FFTs per time step (with one additional forward FFT). Therefore, both methods had the same computational cost. A pseudospectral method would require four real-to-complex FFTs per time step to calculate the two fractional Laplacians in the second-order wave equation (equation 6). However, the SLS model with L relaxation mechanisms would require to solve 3 + L equations in the 2D case or 4 + L equations in the 3D case (Zhu et al., 2013), and it has an effective cost of four real-to-complex FFTs per time step when implemented using a pseudospectral method. However, a pseudospectral implementation poses a strict limit on time step size due to its finite-difference nature. But with one additional forward FFT. Therefore, both methods are approximately the same method.

**CONCLUSIONS**

We introduce a low-rank viscoacoustic wave extrapolation method based on the constant-Q wave equation with decoupled fractional Laplacians. The proposed numerical method can handle an arbitrarily variable fractional power of the wavenumber, and thus it is capable of modeling wave propagation in attenuating media with high accuracy. Using a sign reversal, we formulate a Q-compensated operator for RTM to correct for velocity dispersion and amplitude attenuation at the same time. Synthetic experiments show that the proposed method applied to viscoacoustic data produces subsurface images that are comparable in quality with the results obtained by acoustic RTM applied to acoustic data.

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