Numerical Simulation of Microseismic Data

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We propose an industry sponsored research project to numerically generate microseismic data resulting from hydraulic fracturing operations using a fully elastic 3D multi-component simulation for a suite of sponsor-defined models of Earth’s subsurface.

Proposed start date: January 1, 2014. Duration: 3 years, with annual renewals.

Motivation:

Passive recording of microseismic events generated by subsurface hydrofracturing operations has become a potential tool for monitoring the efficacy of these production-enhancement operations. Specifically, monitoring the generation, location and timing of the induced fractures is used to evaluate the geometric distribution of fracture networks which are presumed to improve the permeability of the subsurface rocks and enhance production from tight shale formations.

Many operators do offer services for monitoring microseismic events and provide the temporal sequences and locations of these events which are associated with the development of the fracture networks. At present, there are no means of independently confirming the location and timing of the ‘interpreted fractures;’ particularly when the subsurface is heterogeneous and anisotropic. Some producers have indicated that providing microseismic recorded data of a particular hydro-fracture monitoring project to different interpretation and processing contractors results in internally consistent and precise locations of fractures; but—each contractor’s interpretation provides a significantly different solution.

Thus, we believe there is a need for ‘ground truth,’ based on known models, to fully evaluate the efficacy of actual microseismic interpretations and to evaluate the sensitivity of interpretations to various subsurface conditions and uncertainties in the interpreted models.
Further, sensitivity to the geometric configuration of the passive recording arrays to estimate fracture event locations can be fully evaluated with these simulations—which allow recording locations anywhere in the three-dimensional volume.

In addition, characterization of the actual fracturing (and faulting) dynamics, expressed as the source mechanism, may lead to improved understanding of the nature of the fractures themselves—providing further information to evaluate the efficacy of the hydrofracking operation.

A sense of current activities in microseismic evaluation methods is provided by a brief perusal of presentations at the most recent annual meeting (October, 2013) of the SEG in Houston. The technical program included seven sessions (Five oral, one E-poster and one traditional poster) of eight presentations each addressing passive seismic methods. The summary in Table I shows the number of presentations in each session to address nine selected topics of interest in microseismic monitoring of hydrofracking operations. (Many papers addressed more than one of these topics.) Note that the primary topics were Event Location (and fracture geometry description) and Source Mechanisms (fracture dynamics). Further, the issues of Velocity Model description, Anisotropy and Receiver Geometry are, perhaps, crucial sub-issues in accurate event location. Receiver Geometry is obviously an issue in both the precision and accuracy of event location as well as describing Source (fracture/fault) Mechanisms.

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<td>1</td>
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<td>P1 Recent Developments</td>
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<tr>
<td>Total:</td>
<td><strong>22</strong></td>
<td><strong>6</strong></td>
<td><strong>8</strong></td>
<td><strong>7</strong></td>
<td><strong>13</strong></td>
<td><strong>5</strong></td>
<td><strong>6</strong></td>
<td><strong>4</strong></td>
<td><strong>2</strong></td>
</tr>
</tbody>
</table>

Table I

This expression of industry focus on issues and its alignment with current problems utilizing microseismic technology provides a clear focus for this proposal and underlying direction for the types of models to be considered.
Proposal:

We propose an industry sponsored project funded by several companies to design 3D fully elastic (21 stiffness components, plus density, at each cell location in the volume) numerical models of the subsurface. Within this volumetric model, we will place sets of microseismic sources (both volumetric and displacement) along predefined fracture locations, and record the seismic arrivals at all locations in the three-dimensional volume. This will provide a complete set of microseismic data for inversion to source locations. Since the source locations are known, a meaningful evaluation of the event location techniques for the models in question and the receiver locations selected is possible. Further, as experience in model building is realized, evaluation of the source characteristics may be realized.

The models will be defined by the sponsors, starting with simple (maybe even a trivial) models, and progress to more complex 3D and anisotropic subsurface and near-surface scenarios. Model description is completely flexible, and receiver response is available at all locations within the model volume. We also propose development of a capability to include user-defined displacements in the description of the microseismic sources. A series of suggested possible models of interest and a projected three-year work flow are included in Appendix I.

Deliverables:

The basic products of the proposed project are the Numerically Simulated Microseismic Data sets from all of the models. Deliverables will include participation in the development of the models and access to all the data. Time snap-shots of all seismic components will be archived and individual participants will be able to select the recording locations and components desired for each of the models. Multiple sponsor requests for various recording geometries from each of the numerical simulations will be possible over the course of the project. Sponsors of the project will have exclusive access to the model results for a two-year period after they are first available to sponsors. After that time, access to the model results will be available in the public domain.

Although the basic products of the projects are the models and simulated data for use by the sponsors in their development and applications areas, the principal investigators will be actively involved in defining the models and assisting the sponsors in the use of the results in their evaluations. Further, the principal investigators will aid in seeking third-party use of the results, such as assisting in evaluation of data by graduate students in their research projects. The project, as proposed, does not include any direct processing or interpretation of the simulated data.
Distribution of results:

Results of the modeling (volumes of vector displacement at each volumetric cell at each time step) will be distributed directly to the sponsors through mutually acceptable means, such as storage on USB disc drives or transfer over the internet. Code to extract time traces for specific recording geometries and recorded components from any desired locations will be included.

Annual meetings and reports will provide summaries and context for the results.

Complete copies of the data sets will be available to the sponsors on request for the cost of the storage media.

Modeling code:

The modeling code available for use by the project remains proprietary to The University of Texas and the original sponsors of the development of the code. The sponsors of the proposed project will be provided with all the implementation details for each model and the results of each simulation. The modeling code is based on the most general elastic seismic wave equation formulation. We use the particle displacement-stress system and all 21 independent elements of the stress tensor. Three components of particle displacement and 6 stress components are computed at all locations for each time step. Spatial derivatives are implemented using the pseudo spectral method to avoid the grid dispersion associated with finite difference operators. The time advances of the wave fronts are computed using the Rapid Expansion Method. The combination of these numerical methods yields very high quality results with little or no numerical artifacts for any type of complex 3D subsurface geometry and anisotropy. Appendix II provides a rather detailed overview of the seismic modeling technique.

Project Structure:

We anticipate several sponsors for a three-year project. The total annual cost is currently budgeted at $543,000. Total cost will be distributed between the sponsors. After the initial year’s commitment continued participation will be by annual renewal.

Members joining after the first year will be required to pay for at least their share of the previous year’s support, as well as the joining year.

Semi-annual operational meetings of Projects PIs and sponsors are planned to define and refine models and monitor results.
The sponsors’ representatives to the project will form the governing board which will be responsible for providing overall guidance to the project, review progress, and will ultimately determine which models are to be run and their source configurations.

Why UT?

- Existing Institutional resources include:
  - Access to existing fully-elastic code.
  - Access to very large parallel computational resources.

- Personnel are available and in place:
  - Faculty with expertise in industry-focused seismology, numerical methods and anisotropy
  - Research scientists and technical staff with expertise in the code and its application

- UT and The UT Institute for Geophysics is a third-party academic organization that can provide standard models for industry wide calibration of interpretation techniques.

Proposed Work for Year 1:

The project work will begin by calibrating the modeling code against 1D reflectivity codes for elastic and VTI media. This study will enable us to assess the numerical accuracy of our pseudo spectral REM modeling method compared to algorithms based on analytical solutions for the special case of 1D structure.

We propose to add the seismic moment tensor source description as another optional way to describe the initial condition. The existing method of defining specific displacements or stresses in space and varying these with time will be maintained and the seismic moment tensor input will simply be an auxiliary way to define the initial condition.

We also need to modify the definition of the initial conditions describing the source positions, initiation time and dynamics to efficiently represent the stress and particle displacements for each grid point and for all time steps. We then must develop the program to extract pressure, stresses and particle displacement components at sponsor specified receiver locations in the models and this capability will be provided to the sponsors with the data distribution.

The first year we will also evaluate preliminary models using multiple sources in time and multiple source types in the simulation of microseismic data. We anticipate the
sponsors will develop 2 to 5 initial models to evaluate during year one. The exact number will be determined in conjunction with the sponsors.

Years two and three will allow construction of more complex models, as well as testing of different subsurface source scenarios. During year 1 we plan to identify, in cooperation with the sponsors, displacement dynamics (geomechanics) associated with the microseismic sources due to hydrofracturing. One possible outcome is a finer scale numerical simulation of this process that can be injected into the existing modeling code as an initial condition or a time varying source function. If this proves viable, during year 2 we will develop, implement and test this approach as an additional source type or initial condition.

A summary of Possible Models and Issues to be addressed is included in Appendix I. This appendix also includes a brief summary of a possible Working Schedule projected for the proposed three-year project.

Roles of the Senior Personnel
Tatham will work with the sponsors to define the problems to be addressed by the modeling. Jointly he and the sponsors will develop the models to be evaluated and provide basic interpretations of the data generated.

Stoffa will work with Tatham to finalize the models and to insure they can be computed both accurately and in a timely fashion. He will work with Seif to define the numerical requirements for each model, to run the model, to document the model run and to evaluate the results. All model seismic data will be reviewed for quality and then archived.

Seif will work with Stoffa to run the models and to add new capabilities to the modeling code as needed. He will write the data extraction program to gather specific receiver geometries from the data volumes. He will also add new source mechanisms as developed by Sen.

Sen will provide interpretation of the model seismograms and start research into how to incorporate more complicated in space and time source modeling into the micro seismic source mechanism. These new source mechanisms will be available for modeling runs in year 2 of the project. He will continue to investigate how geomechanics can be used to refine the source mechanism and to add this capability to the modeling in the later years of the project.
Preface to the Budget

The budget only includes the cost for the principals to participate with the sponsors in the model design, numerical simulation and preliminary evaluation of the results. Funds are not requested for detailed processing and interpretation of the data generated. We assume that the sponsors will do their individual interpretations and evaluations using their own proprietary algorithms and software.

Because we limit our study to the numerical simulations and not the detailed interpretations of the results, we fully expect the data to be used by graduate students in the Jackson School of Geosciences at UT Austin and possibly elsewhere. In particular, project EDGER in the Department of Geological Sciences of the Jackson School is an Industry sponsored consortium which sponsors graduate students doing research related to the models we will test by numerical simulation. These students will benefit directly from this sponsored research as the data will be made available to them at the same time or shortly after it is made available to the sponsors.

Proposed Budget

01 January 2014 - 31 December 2014

<table>
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<tr>
<th>Description</th>
<th>Amount</th>
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<tr>
<td>Salaries and Fringe Benefits</td>
<td>$283,215</td>
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<tr>
<td>Travel</td>
<td>13,536</td>
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<tr>
<td>Other Direct Costs</td>
<td>54,705</td>
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<tr>
<td><strong>Total Direct Costs</strong></td>
<td><strong>351,456</strong></td>
</tr>
<tr>
<td>Total Indirect Costs</td>
<td>54.5% MTDC</td>
</tr>
<tr>
<td><strong>Total Direct and Indirect Costs</strong></td>
<td><strong>$543,000</strong></td>
</tr>
</tbody>
</table>

Cost per sponsor, based on ten sponsors, is $54,300.

Next Steps:

Contract commitments from each sponsor have to occur during the last quarter of 2013, with the project commencing January 1, 2014.
Contact Information:  Project Principal Investigators:

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Mrinal Sen  mrinal@utig.ig.utexas.edu

Coordinators:  Rosalind Gamble
             Margo Grace
Appendix I

Suggested Models and possible Implementation schedule

Possible Models and Issues to Address

The models to be developed will be defined by the sponsors, in coordination with the principal investigators to optimize the objectives to be addressed and the actual modeling capability. As a starting point, a possible path for both Issues to address and Model building is included below. One approach is to consider some of the issues at hand, then let the development of the models to address these issues evolve as experience is gained and more complex models can effectively address individual issues.

Issues:

- Uncertainties in the location of microseismic events
  - Velocity field definition
    - Inhomogeneities
      - Shale properties
      - Overburden conditions
    - Recording Geometries
      - Surface Arrays
      - Borehole arrays

- Fracture/Fault Plane Geometries events location and timing
  - Interference from closely space events in time
  - Mapping of event sequences

- Definition of Fracture/Fault Displacement Characteristics
  - How well can we define them from microseismic observations?
  - Recording geometries required
    - Azimuthal distribution
    - Borehole vs Surface

- Subsurface rock Characterization
  - Can we describe rock parameters from microseismic observations?
    - Anisotropy
    - Thin Layer geometry
Possible Models:

- Source Location Testing
  (Receiver locations are available at ALL cells in the model)
  - Surface
  - Borehole
    - Vertical
    - Horizontal
    - In shale layer
  - Homogeneous/Isotropic Medium
  - Variable shale Isotropic Medium
  - Anisotropic Shale
  - More complex shale systems
  - More complex models

Analyze microseismic response for various receiver configurations

- Repeat sequence with multiple sources on a fracture/fault surface
- Extend to different source mechanism, both single sources and multiple sources (with varying source mechanisms?) on a fracture/fault surface

- Velocity Field Variations
  - Extend simple models to include increasingly complex velocity models
    - Overburden considerations
      - Near Surface
      - Entire overburden above the shale
      - IntraShale velocity effects
        - Anisotropy
        - Thin Layers

- Source Mechanism Variations
  - Response to a single source at a single position with various source mechanism
  - Response to multiples sources on a fracture/fault plane
    - Single Source Mechanism for multiple sources
    - Multiple sources with various source mechanisms on a fracture/fault surface
Projected work and deliverable schedule

<table>
<thead>
<tr>
<th>Year</th>
<th>Model Building</th>
<th>Modeling Capability</th>
<th>Models Delivered</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Single source models</td>
<td>Adapt initial conditions to permit multiple sources. Code extract microseismic traces from models.</td>
<td>Preliminary single-source models (User selectable receiver positions)</td>
</tr>
<tr>
<td></td>
<td>Multiple Source Models along user-defined fracture/fault surfaces</td>
<td>Implement multisource capability. Begin work on user-defined displacement for sources (Moment Tensor Sources)</td>
<td>More complex models, including preliminary multiple source models</td>
</tr>
<tr>
<td>2</td>
<td>Include larger, more complex models. Expand modeling with user-defined sources and displacement.</td>
<td>Continue development and implementation of user-defined source mechanisms.</td>
<td>Preliminary user-defined source mechanisms. More complex fracture geometry descriptions.</td>
</tr>
<tr>
<td></td>
<td>Include both user-defined source geometry and mechanism models</td>
<td>Preliminary sequences of sources with variable source mechanisms.</td>
<td>Complete modifications for User-defined source location &amp; mechanisms.</td>
</tr>
<tr>
<td>3</td>
<td>Define more complex models, including ‘standard’ models for industry-wide calibration.</td>
<td>Flexibility in source positioning &amp; fault/fracture dynamics.</td>
<td>Multiple user-defined models, including ‘standard’ models for testing processing and interpretation capability.</td>
</tr>
<tr>
<td>3+</td>
<td>Continue project after three years? Include testing geomechanical issues with numerically generated models?</td>
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Appendix II

Parallel elastic 3D modeling program
developed by:
Paul L. Stoffa, Roustam Seif and Mrinal Sen
with the support of ConocoPhillips
Appendix

Models will be developed by sponsors to evaluate the seismic response for different subsurface models.

We will use a parallel elastic 3D modeling program developed by:

Paul L. Stoffa, Roustam Seif, Mrinal Sen

with the support of ConocoPhillips
Overview

One parallel program for all simulations: acoustic, elastic, vti, tti, hti

3D elastic seismic modeling
  for arbitrary $C_{ij}$
  pseudo spectral in space
  rapid expansion method (REM) in time
3D Elastic Modeling

We solve the particle displacement-stress equations using a pseudo spectral method for all spatial derivatives and the rapid expansion method for the explicit time stepping.

This combination of methods is unconditionally numerically stable and accurate up to the maximum wave numbers of the grid.

We allow the $C_{ij}$ to be completely arbitrary and particular special cases (e.g. vti, tti) will be pre mapped into the $C_{ij}$ for computational purposes.

The numerical scheme is inherently parallel and requires 9 banks of nodes, each bank may actually contain 10's of nodes to do the work involved.
3D elastic wave equation

3D elastic wave equation in anisotropic media can be expressed in displacement-stress format

\[
\begin{align*}
\rho \frac{\partial^2 u_x}{\partial t^2} &= \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} \\
\rho \frac{\partial^2 u_y}{\partial t^2} &= \tau_{xy,x} + \tau_{yy,y} + \tau_{yz,z} \\
\rho \frac{\partial^2 u_z}{\partial t^2} &= \tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z}
\end{align*}
\]

where \( u \)'s and \( \tau \)'s represent respectively particle displacements and stresses, \( \rho \) is the density, \( C_{ij} \) are stiffness tensor components, and subscripts after the comma stand for differentiation.
3D elastic wave equation

We write the two systems as operators, \( G_1 \) and \( G_2 \)

\[
\frac{\partial^2}{\partial t^2} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} \\ \tau_{xy,x} + \tau_{yy,y} + \tau_{yz,z} \\ \tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z} \end{pmatrix} = G_1(\tau)
\]

\[
\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} u_{x,x} \\ u_{y,y} \\ u_{z,z} \\ u_{y,z} + u_{z,y} \\ u_{x,z} + u_{z,x} \\ u_{x,y} + u_{y,x} \end{pmatrix} = G_2(u)
\]
Numerical scheme: space

Spatial derivatives using the pseudo spectral method

\[
\frac{\partial f}{\partial x} = \text{IFFT} \left[ ik_x e^{\pm ik_x \frac{\Delta x}{2}} \text{FFT} [f(x)] \right]
\]

3D FFT for derivatives:
9 for stresses
9 for displacements

as a consequence, we use 9 groups of nodes
Each group has multiple nodes & the data volume is distributed in y
3D FFTs are done in parallel within each group
Derivatives are done in parallel by each group
Next $\Delta t$

$C_{ij}$ multiply

Group 1
\[ \frac{\partial \tau_{xx}}{\partial x} \]

Group 2
\[ \frac{\partial \tau_{yy}}{\partial y} \]

Group 3
\[ \frac{\partial \tau_{zz}}{\partial z} \]

Group 4
\[ \frac{\partial \tau_{yz}}{\partial z} \]

Group 5
\[ \frac{\partial \tau_{zx}}{\partial z} \]

Group 6
\[ \frac{\partial \tau_{xy}}{\partial y} \]

Group 7
\[ \frac{\partial \tau_{yz}}{\partial y} \]

Group 8
\[ \frac{\partial \tau_{zx}}{\partial y} \]

Group 9
\[ \frac{\partial \tau_{xy}}{\partial y} \]
reuse the code

The same subroutine can be used for all derivative computations

\[
\frac{\partial^2}{\partial t^2} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \frac{1}{\rho} I_3 \begin{pmatrix} \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} \\ \tau_{xy,y} + \tau_{yy,y} + \tau_{yz,z} \\ \tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z} \end{pmatrix}
\]
Displacement components

ux & ux,x
uy & uy,y
uz & uz,z
uy & uy,z
ux & ux,z
ux & ux,y
uz & uz,x
uz & uz,y
uy & uy,x
Stress components

1. tx & tx,x
2. ty & ty,y
3. tyy & tyy,y
4. txz & txz,z
5. tyy & tyy,y
6. txy & txy,y
7. tyz & ty,y
8. txz & txz,x
9. txy & txy,x
Rapid Expansion Method: time

Introducing vector notation

\[ \vec{u} = (u_x, u_y, u_z) \]

system may be rewritten as

\[ \frac{\partial^2 \vec{u}}{\partial t^2} = -G^2 \vec{u}, \text{ where } -G^2 = G_1G_2 \]

Now use the REM scheme

\[ \vec{u}_t = -\vec{u}_{-t} + 2 \cos(Gt)\vec{u}_0 \]

\[ \Rightarrow \vec{u}_t = 2 \sum_{k=0}^{\infty} C_k J_k (tR) Q_k \left( \frac{iG}{R} \right) \vec{u}_0 \]

\[ R = \max(v_p) \pi \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}} \]
Work flow

Update time derivatives

\[
\frac{\partial^2}{\partial t^2} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \frac{1}{\rho} I_3 \begin{pmatrix} \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} \\ \tau_{xy,x} + \tau_{yy,y} + \tau_{yz,z} \\ \tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z} \end{pmatrix}
\]

Multiply volumes

\[
\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix}
\]

Compute derivatives

\[
\begin{pmatrix} u_{x,x} \\ u_{y,y} \\ u_{z,z} \\ u_{y,z} + u_{z,y} \\ u_{x,z} + u_{z,x} \\ u_{x,y} + u_{y,x} \end{pmatrix}
\]
For all problems we map the ‘geophysical’ model description into the $C_{ij}$ matrix

\[ \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{x,x} \\ u_{y,y} \\ u_{z,z} \\ u_{y,z} + u_{z,y} \\ u_{x,z} + u_{z,x} \\ u_{x,y} + u_{y,x} \end{pmatrix} \]

hence

\[ \tau_{xx} = \tau_{yy} = \tau_{zz} = \lambda (u_{x,x} + u_{y,y} + u_{z,z}) = P, \quad \frac{1}{\lambda} P = u_{x,x} + u_{y,y} + u_{z,z} \]

\[ \tau_{yz} = \tau_{xz} = \tau_{xy} = 0 \]
For all problems we map the ‘geophysical’ model description into the $C_{ij}$ matrix

**Elastic isotropic**

In this case the $C_{ij}$ matrix becomes

$$
\begin{pmatrix}
C_{33} & (C_{33} - 2C_{44}) & (C_{33} - 2C_{44}) & 0 & 0 & 0 \\
(C_{33} - 2C_{44}) & C_{33} & (C_{33} - 2C_{44}) & 0 & 0 & 0 \\
(C_{33} - 2C_{44}) & (C_{33} - 2C_{44}) & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44} \\
0 & 0 & 0 & 0 & 0 & 0 & C_{44}
\end{pmatrix}
$$

Taking into account that $C_{33} = \lambda + 2\mu$ and $C_{44} = \mu$

$$C_{33} = \rho v_p^2 = \lambda + 2\mu, \quad \text{and} \quad C_{44} = \rho v_s^2 = \mu, \quad \text{so}$$

$$\lambda = \rho (v_p^2 - 2v_s^2)$$
For all problems we map the ‘geophysical’ model description into the $C_{ij}$ matrix.

**Elastic isotropic**

Which we recognize as:

$$
\begin{pmatrix}
\tau_{xx} \\
\tau_{yy} \\
\tau_{zz} \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{pmatrix}
= 
\begin{pmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_{x,x} \\
\mathbf{u}_{y,y} \\
\mathbf{u}_{z,z} \\
\mathbf{u}_{y,z} \ + \mathbf{u}_{z,y} \\
\mathbf{u}_{x,z} \ + \mathbf{u}_{z,x} \\
\mathbf{u}_{x,y} \ + \mathbf{u}_{y,x}
\end{pmatrix}
$$

\begin{align*}
\lambda + 2\mu &= \rho v_p^2 \\
\lambda &= \rho (v_p^2 - 2v_s^2) \\
\mu &= \rho v_s^2
\end{align*}
For all problems we map the ‘geophysical’ model description into the $C_{ij}$ matrix

In this case the $C_{ij}$ matrix becomes

$$
\begin{pmatrix}
C_{11} & (C_{11} - 2C_{66}) & C_{13} & 0 & 0 & 0 \\
(C_{11} - 2C_{66}) & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{pmatrix}
$$

Here

$$
C_{33} = \rho v_p^2 ,
$$

$$
C_{44} = \rho v_s^2 ,
$$

$$
C_{11} = (1 + 2\varepsilon)C_{33} ,
$$

$$
C_{66} = (1 + 2\gamma)C_{44},
$$

$$
C_{13} = -C_{44} + \sqrt{(C_{33} - C_{44})^2 + 2\delta C_{33}(C_{33} - C_{44})}
$$
TTI and HTI

• TTI : rotated VTI
• HTI : special case of TTI
• Spatial rotation is described by 3x3 matrix R
• Transformation of 6x6 stiffness Cij matrix is described by formula

\[ \text{Cij\_transformed} = B \cdot \text{Cij} \cdot B^t \]

• Here 6x6 Bond matrix computed from R
Rotations

\[
\begin{align*}
\theta_{y'y} &= \cos^{-1} a_{yy} \\
\theta_{y'x} &= \cos^{-1} a_{yx} \\
\theta_{x'y} &= \cos^{-1} a_{xy} \\
\theta_{x'x} &= \cos^{-1} a_{xx} = \xi
\end{align*}
\]
Rotation matrix $R$

$$R = \begin{pmatrix}
a_{xx} & a_{xy} & a_{xz} \\
a_{yx} & a_{yy} & a_{yz} \\
a_{zx} & a_{zy} & a_{zz}
\end{pmatrix}$$
Bond Matrix B

\[
\begin{bmatrix}
a_{xx}^2 & a_{xy}^2 & a_{xz}^2 \\
a_{yx}^2 & a_{yy}^2 & a_{yz}^2 \\
a_{zx}^2 & a_{zy}^2 & a_{zz}^2 \\
\end{bmatrix}
\begin{bmatrix}
2a_{xy}a_{xz} & 2a_{xz}a_{xx} & 2a_{xx}a_{xy} \\
2a_{yy}a_{yz} & 2a_{yz}a_{yx} & 2a_{yx}a_{yy} \\
2a_{zy}a_{zz} & 2a_{zz}a_{zx} & 2a_{zx}a_{zy} \\
\end{bmatrix}
\begin{bmatrix}
a_{yx}a_{zx} & a_{yy}a_{zy} & a_{yz}a_{zz} + a_{yza_{zz}} \\
a_{zx}a_{xx} & a_{zy}a_{xy} & a_{xz}a_{zz} + a_{xza_{zz}} \\
a_{xx}a_{yx} & a_{yy}a_{yz} & a_{xy}a_{yx} + a_{xy}a_{xz} \\
\end{bmatrix}
\]

Cij\_transformed = B*Cij*B\textsuperscript{t}
\[
\begin{align*}
\text{VTI} & \quad \text{HTI} \\
C_{11} & (C_{11} - 2C_{66}) & C_{13} & 0 & 0 & 0 \\
(C_{11} - 2C_{66}) & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{align*}
\]

\[
\begin{align*}
\text{TTI} & \quad \text{HTI} \\
(C_{11} - 2C_{66}) & (C_{11} - 2C_{66}) & C_{11} & 0 & 0 & 0 \\
C_{13} & C_{11} & (C_{11} - 2C_{66}) & 0 & 0 & 0 \\
C_{13} & (C_{11} - 2C_{66}) & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{align*}
\]
For all problems we map the ‘geophysical’ model description into the $C_{ij}$ matrix

In this case the $C_{ij}$ matrix becomes

$$
\begin{pmatrix}
C_{33} & C_{13} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{11} & (C_{11} - 2C_{66}) & 0 & 0 & 0 \\
C_{13} & (C_{11} - 2C_{66}) & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}
\end{pmatrix}
$$

Here

$C_{33} = \rho v_p^2$, \\
$C_{44} = \rho v_s^2$, \\
$C_{11} = (1 + 2\varepsilon)C_{33}$, \\
$C_{66} = (1 + 2\gamma)C_{44}$, \\
$C_{13} = -C_{44} + \sqrt{(C_{33} - C_{44})^2 + 2\delta C_{33}(C_{33} - C_{44})}$
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