Moist air thermodynamics
(Reading Text 3.5-3.7, p79-101)

Topics:
• Variables that descript moist air
• static stability for moist convection, saturated and pseudo-adiabatic lapse rates, equivalent potential temperature
• Static stability
• The second law of thermodynamics for moist air, the Clausius-Clapeyron equation
Variables

• Mixing ratio, $w$, and specific humidity, $q$:
  – $W = \frac{m_v}{m_d}$, mass of vapor vs. mass of dry air
  – $q = \frac{m_v}{(m_v + m_d)} = \frac{w}{(w+1)}$, mass of vapor vs. total air mass

• Vapor pressure, $e = \frac{w}{(w+\epsilon)}p$
  – Where recall $\epsilon = \frac{M_v}{M_d} = \frac{R_d}{R_v} = 0.622$, $p$: air pressure
    Because:
    $$e = \frac{n_v}{n_v + n_d} p = \frac{\frac{m_v}{M_v}}{\frac{m_d}{M_d} + \frac{m_v}{M_v}} p \times \frac{\frac{M_v}{m_d}}{\frac{M_v}{m_d} + \frac{m_v}{m_d}} = \frac{m_v}{\frac{M_v}{m_d} + \frac{m_v}{M_v}} p = \frac{w}{\epsilon + w} p$$

• $n_v, n_d$: numbers of mole of vapor and dry air molecules, respectively. $m_v/m_d$, mass of vapor
• $M_v$ and $M_d$, molecular weight of vapor and dry air
Virtual temperature $T_v$:

- Recall from Sect. 4.1, that $T_v = T/[1-e/p(1-\epsilon)]$ (Eq. 3.16 in the text), we can express $T_v \approx T (1 + 0.61W)$ (see below)

\[
e = \frac{w}{\epsilon + w} \quad p \Rightarrow \quad \frac{e}{p} = \frac{w}{\epsilon + w}
\]

\[
T_v = T \frac{1}{1 - \frac{e}{p}(1 - \epsilon)} = T \frac{1}{1 - \frac{w}{\epsilon + w}(1 - \epsilon)} = T \frac{\epsilon + w}{\epsilon + w - w + w\epsilon}
\]

\[
T_v = T \frac{1 + \frac{w}{\epsilon}}{(1 + w)} = T \left(1 + \left(\frac{1}{0.622} - 1\right) \frac{w}{1+w}\right) = T \left(1 + 0.61 \frac{w}{1+w}\right)
\]

$T_v \approx T(1 + 0.61w)$ because $w \ll 1$, $w + 1 \approx w$
Question:
If the atmospheric consists 10 g per 1 kg of dry air and temperature is 27°C at the sea-level, what is the mixing ratio (w), specific humidity (q), vapor pressure (e) and virtual temperature of this atmosphere?
• \( W = \frac{10\text{g}}{1\ \text{kg}} = 0.01 \)

• \( q = \frac{10\text{g}}{1000\text{g} + 10\text{g}} = 0.0099 = 9.9\text{g/kg} \)

• \( e = \frac{wP}{(\varepsilon + w)} = \frac{0.01 \times 1000\text{hPa}}{0.622 + 0.01} = 15.8\text{Pa} \)

• \( T_v \approx T(1 + 0.61w) = (273 + 27)K(1 + 0.61 \times 0.01) = 301.8K = 28.8\degree C \)
• **Saturation vapor pressure, \( e_s \):** the maximum vapor pressure with respect to a plane surface of pure water at \( T \). It is a function of \( T \).

• Below 0°C, the saturation vapor pressure with respect to ice, \( e_{si} \), for a plane ice surface is lower than that with respect of water.

• Thus, atmosphere can be saturated for ice but not for water. Supersaturation \( e > e_{si} \) can occur in real atmosphere (e.g., near tropopause).
• Saturation mixing ratio, \( w_s = m_{vs}/m_d = 0.622e_s/(p-e_s) \approx 0.622e_s/p \)
  – Saturation mixing ratio is more commonly used in meteorology and climate than \( e_s \).
  – See p. 82 of the text for derivation.

• Relative humidity, \( RH = 100w/w_s = 100e/e_s \)
  – Relative humidity is derived from dew point or frost point temperature, \( T_d \), and air temperature, \( T \), which can be measured readily.

• \( T_d \): the temperature at which dew starts to form, when \( w_s(T_d) = w \) of the air, thus \( RH = 100w_s(T_d)/w_s(T) \)
  Thus, RH is determined by saturation mixing ratio of dew/ frost point temperature vs. that of air temperature for the same pressure value.
Example:

- Meteorological measurements show that a) air $T=30^\circ C$ and $T_d=25^\circ C$; b) The next day, $T$ remains the same, but $T_d$ reduces to $20^\circ C$. Calculate RH for both cases use the figure below:

a) $e_s(T=30^\circ C)=40$ hPa,
   
   $e_s(T_d=25^\circ C)=30$ hPa,
   
   $RH_a=100 \times \frac{e_s(T_d)}{e_s(T)}=75\%$

b) $e_s(T_d=20^\circ C)=25$ hPa
   
   $RH_b=100 \times \frac{e_s(T_d)}{e_s(T)}=25\text{hPa}/40\text{hPa}=62.5\%$

   Relative humidity dropped about $12.5\%$ in case b)
• Lift condensation level (LCL):
  
  – T in a rising air parcel would decrease follow the dry adiabatic lapse rate (9.8°C/km). At the height where T reduces to the same value as $T_d$, condensation occurs. This height is referred to as the lifting condensation level (LCL). It is also cloud base.
• **LCL in a skew T-\(\ln p\) chart:**

- **T** in a rising air parcel would follow constant \(\theta\) line (orange, dry adiabatic line). The green isothermal lines indicate \(T\) and \(T_d\) of the surface air (1000 hPa). The pressure level at which constant \(\theta\) line cross the isothermal line for \(T_d\) is the LCL for this air.

![Figure 3.10 The lifting condensation level of a parcel of air at A, with pressure \(p\), temperature \(T\) and dew point \(T_d\), is at C on the skew T – \(\ln p\) chart.](image)

Figure 3.10 The lifting condensation level of a parcel of air at A, with pressure \(p\), temperature \(T\) and dew point \(T_d\), is at C on the skew T – \(\ln p\) chart.
Saturation moist adiabatic and pseudoadiabatic lapse rates:

- As an rising air parcel above LCL, condensation occurs and the temperature change inside of the air parcel is determined by combined dry adiabatic cooling and latent heating due to condensation \( \frac{dT}{dz} = \left( \frac{\alpha dp}{dz} - L_{dw} \right)/C_p \), vs. dry adiabatic \( \frac{dT}{dz} = \frac{\alpha dp}{(C_p dz)} \). Thus, the rate of temperature decrease with height, i.e., the moist adiabatic lapse rate \( (\Gamma_m) \), is less than that of dry adiabatic lapse rate.

- Unlike the dry adiabatic lapse rate \( (\Gamma_d = R/C_p = 9.8 K/km) \), \( \Gamma_m \) is not a constant. \( \Gamma_m \) depends on net amount of condensation.

- Two assumptions about \( \Gamma_m \):
  - **Saturation moist adiabatic lapse rate**: All condensed water is retained in the rising air parcel, and can be re-evaporated if \( T \) increases. It is a reversible process.
  - **Pseudoadiabatic lapse rate**: All condensed water falls out the rising air parcel. Thus, re-evaporated is not possible. It is a irreversible process.
Saturation moist adiabatic vs. pseudoadiabatic lapse rates:

• Saturated moist adiabatic lapse rate is greater than pseudoadiabatic lapse rate (T decreases faster with height), because
  – In case of the former, condensed water can be re-evaporated which absorbs latent heat. Thus, air parcel tends to be less buoyant and $T_v$ (virtual temperature) decreases faster with height.
  – Under pseudoadiabatic case, no re-evaporation of condensed water. The rising air parcel is more buoyant and $T_v$ decreases slower with height.

• Moist lapse rate varies. Typically, $\Gamma_m$ is about 4 K/km in the warm and humid lower troposphere, and ~ 6-7 K/km in the mid-troposphere, close to $\Gamma_d$ in the upper troposphere. Why?
Example:

Meteorological measurements show that air $T=30^\circ C$ and $T_d=25^\circ C$, $T=-5^\circ C$ at 500 hPa.

a) Determine the LCL of an rising air parcel from the surface

b) Determine the lapse rate between the LCL and 500 hPa assuming i) all condensed liquid water falls out immediately; ii) only a half of the liquid fall out, the other half rising with the air parcel and re-evaporate immediately. Ignore the heat absorbed by the condensed water.

c) Height of 500 hPa is 5.5 km, 950hPa is 0.5 km. Latent heat of vaporization, $L_v=2.5\times10^6$ J kg$^{-1}$. Surface pressure is 1000 hPa.
Solution:

a. From the surface to LCL, \[ \frac{\partial T}{\partial z} = \Gamma_d = 9.80 \text{K/km}, \quad T - T_d = \frac{\partial T}{\partial z} z_{LCL} \]

\[ z_{LCL} = \frac{T - T_d}{\Gamma_d} = \frac{(30 - 25)K}{9.8 \text{K/km}} \approx 0.51 \text{km} \]

b. At LCL, \( e_{s,LCL} = e_s(25C) = 30 \text{hPa} \), \( w_s = 0.622 \frac{e_s}{p} = 0.622 \frac{30 \text{hPa}}{950 \text{hPa}} = 0.020 = 20 \text{g/kg} \)

At 500 hPa, \( e_{s,500\text{hPa}}(-5C) = 5 \text{hPa} \), based on \( e_s - T \) relation shown in the figure.

\[ w_{s,500\text{hPa}} = 0.622 \frac{e_s}{p} = 0.622 \frac{5 \text{hPa}}{500 \text{hPa}} = 6.2 \times 10^{-3} = 6.2 \text{g/kg} \]

i) The total condensed water between LCL and 500 hPa, \( \delta e_s = (20 - 6.2) \text{g/kg} = 13.8 \text{g/kg} \)

\[ \frac{dT}{dz} = \frac{1}{C_p} \left( \alpha \frac{dp}{dz} - \frac{L}{C_p} \frac{dw}{dz} \right) = \frac{g}{C_p} \frac{L}{C_p} \frac{dw}{dz} = \frac{9.8 \text{K/km}}{1004 \text{JK}^{-1} \text{kg}^{-1}} \cdot \frac{13.8 \times 10^{-3}}{5 \text{km} \times 10^3 \text{m/km}} \]

\[ = (9.8 - 6.9) \text{K/km} \approx 2.9 \text{K/km} \]

ii) If a half of the condensed water re-evaporates, the net condensed water is 6.9 g/kg.

\[ \frac{dT}{dz} = (9.8 - 3.45) / 1004 \text{K/m} \approx 0.00635 \text{K/m} \]

Thus, the value of moist lapse rate depends on the amount of net condensation in the rising air parcel.
• Wet-bulb temperature, $T_w$:
  – In surface meteorological station, web-bulb temperature represents $T$ measured by a glass bulb thermometer wrapped by wet clothe over which ambient air is drawn.
  – Because evaporation of wet clothe also cool ambient $T$ and increase $w$, thus $T_w > T_d$ (cool dry adiabatically until condensation occurs) $< T$, usually is the arithmetic mean of $T$ and $T_d$. 
Equivalent potential temperature, $\theta_e$: 

- When condensation occurs in a rising air parcel, the latent heat released by condensation, $L_vdw_s$, would warm the air parcel, thus against the dry adiabatic cooling, as shown by the 2\textsuperscript{nd} law of thermodynamics $L_vdw_s=C_pdT-\alpha dp$.

- The equivalent potential temperature, $\theta_e$, is defined as the temperature would be if all the water vapor were condensed and rain out immediately (pseudoadiabatic moist process), and all the potential energy were used to heat the air. Thus, it is higher than the potential temperature $\theta$ (only include internal and potential energy, not latent energy).

- $\theta_e$ is derived from the 2\textsuperscript{nd} law of thermodynamics $-L_vdw_s=C_pdT-\alpha dp$ assuming all the water vapor is condensed and rain out immediately.
because $\theta = T\left(\frac{P_o}{p}\right)^{R/c_p}$, $\ln\theta = \ln T - \frac{R}{C_p} \ln p + \frac{R}{C_p} \ln p_o$

$C_p \frac{d\theta}{\theta} = C_p \frac{dT}{T} - R \frac{dp}{p}$, because $\frac{dq}{T} = C_p \frac{dT}{T} - R \frac{dp}{p}$

\[
\frac{dq}{T} = C_p \frac{d\theta}{\theta} \Rightarrow L_v dw_s = C_p \frac{d\theta}{\theta} \Rightarrow
\]

\[
\frac{d\theta}{\theta} = -\frac{L_v dw_s}{C_p T} \approx -d\left(\frac{L_v w_s}{C_p T}\right) \text{ because } \frac{1}{T^2} \ll \frac{1}{T}
\]

\[
\int_{\theta_e}^{\theta} \frac{d\theta}{\theta} = -\int_0^{w_s} d\left(\frac{L_v w_s}{C_p T}\right)
\]

\[
\theta_e \approx \theta \cdot \exp\left(\frac{L_v w_s}{C_p T}\right)
\]

$\theta_e$ is defined as the equivalent potential temperature, represents the potential temperature ($\theta$) would be if all the vapor were condensed and its latent heat were used to increase $\theta$. 
• Equivalent potential temperature, $\theta_e$ is a conserved variable for an pseudoadiabatic adiabatic process, i.e., there is no heat exchange between the system and the ambient, and 100% latent heat release due to condensation of water vapor is used to increase T.

• The moist adiabatic process is not a true adiabatic process, because of latent heat release. However, we can treat it as it were a adiabatic process if we use $\theta_e$ concept.

• $\Theta_e$ is similar to $\theta$ for dry air.
Normand’s rule-finding $T_d$ and $T_w$ using $\theta$ and $\theta_e$ lines

- Find LCL:
  - Find the $\theta$ line that intersects with isothermal line that represents surface $T$.
  - Find the $w_s$ line that intersects with $T_d$ at 1000 hPa.
  - The intersection between $\theta$ and $w_s$ represents LCL.

- Finding $T_w$ in Skew T-lnp chart:
  - From the LCL follow the constant $\theta_e$ line down to the sea level (1000 hPa), you will find the wet bulb temperature, $T_w$ at the surface.

- Q: Why does LCL is determined by the $T_d$ follow the $W_s$ instead of isothermal line?
Effect of ascent followed and descent on air temperature and humidity:

• Once a convective air reaches LCL (point 2), it would ascent follow constant $\theta_e$ line until it reaches cloud top (point 3) and begin descending.

• The temperature of the descending air depends on the type of moist process during ascending process:
  - If it follows pseudoadiabatic process, all the condensed water falls out, no evaporation would occur during the descending, T follows dry adiabatic process (constant $\theta$ line) back to surface (point 4, 900 hPa in this case) with T₂ (see exercise 3.10 for quantitative description)
  - If it follows saturated moist process, condensed water retained in the air parcel would re-evaporate, T during descending follows $\theta_e$ to LCL (point 2), then follow $\theta$ back to surface (point 1). Thus, T is reversal.
In-class discussion-summary:

• Specific humidity is more commonly used in meteorological application. Do you expect the value of specific humidity more or less than that of mixing ratio? Why?
• How is RH determined by meteorological measurement?
• What is the difference between the saturation moist adiabatic and pseudoadiabatic moist assumptions? Which assumption would lead to overestimate buoyancy of the convecting air (or strength of convection) and which assumption would underestimate the buoyancy of convective air? Why?
• What determine value of the moist adiabatic lapse rate?
• What type of lapse rate does equivalent potential temperature represent?