A NOVEL EXTENDED HYBRID BORN-RYTOV (HBR) FRAMEWORK FOR USE IN ISOTROPIC AND ANISOTROPIC AVO ANALYSIS

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ABSTRACT
In this paper, we will: 1) describe how the HBR theory can be extended to allow use in anisotropic elastic media; and, 2) propose a further extension to the HBR framework that expands its valid range to include both “sub-Born” and “post-Rytov” scenarios. In addition to providing the theoretical foundation for a generalized elastic anisotropic HBR framework, we will outline methods to: a) easily produce HBR AVO approximations ($R_{pp}$) for any anisotropy level by applying a straightforward correction to the well-known Born reflectivity solutions (e.g., the 3-term Aki-Richards AVO approximation in isotropic media); and, b) automatically determine the ideal value for the tuning parameter ($n$) as a function of the physical/elastic parameters of the media generating the reflection and the seismic incident angle. This tuning parameter controls whether the HBR solution is in the “sub-Born”, “Born”, “Born-Rytov”, “Rytov”, or “post-Rytov” regimes. Initial results indicate that the “background trend” for most rocks is a Born-like solution at normal incidence and a gradual transition to [post-]Rytov solutions as incident angle increases. We have formulated our methodology such that it can be directly applied to any type of elastic anisotropy without needing modification – all anisotropic effects are accounted for by choosing an appropriate Born-based reflectivity expression ($R_{pp}^{\text{Born}}$). Lastly, we present some initial work that investigates the usefulness of $n$ as an indicator of subsurface properties. We implemented our preliminary tests using synthetic data generated from a real suite of well logs from the North Sea with 9 identified facies (3 shales, 3 brine sands, and 3 oil sands). Our initial focus has been on enhancing human-based classification and interpretation, and while more testing is needed to show conclusive results our initial results do suggest that there are some easily identifiable features in plots comparing $n$ vs. angle ($\theta$) and depth ($d$) that might provide useful about subsurface properties of interest. For example: our initial results suggest that a) finely interbedded packages produce an anomalous sub-Born signal, and b) changing pore fluid from brine to oil seems to “slow down” the transition from Born ($\theta = 0^\circ$) to [post-]Rytov ($\theta \gg 0^\circ$) by 5-15°. In future research, we plan to continue evaluating the classification potential of $n$ in human-based classification/interpretation as well as in both statistical classification methods (e.g., Probabilistic k-nn) and in seismic inverse problems.
INTRODUCTION

Understanding the scattering phenomenon of elastic seismic waves plays an important part in predicting the seismic response of any given system. Scattering can be used to predict diffractions produced by point diffractors as well as reflections produced by coherent structures in the subsurface. As such, seismic scattering plays an important role in both forward modeling of seismic wave propagation and, by extension, in solving inverse problems related to seismic wave propagation such as seismic migration and seismic inversion.

The exact science behind understanding seismic scattering and seismic reflectivity is well known. Unfortunately, the precise equations that describe these phenomena are inconvenient and unintuitive to work with. During the 1980’s, when AVO first became prevalent, utilizing the exact reflectivity expressions also presented a non-trivial computation expense in large scale projects, though today that expense pales in comparison to processing workflows such as Full Waveform Inversion (FWI) that require multiple wave-equation based modeling operations. In general, the equations describing seismic reflectivity can be derived by considering wave propagation at a layer interface and setting boundary conditions to ensure stress and displacement are continuous across the interface (e.g., Graebner, 1992, describes this for VTI media). In isotropic media these equations converge to the well-known Zoeppritz equations (Zoeppritz, 1919). In anisotropic media, one could technically derive Zoeppritz-like solutions (this involves a symbolic Eigen-decomposition of the Christoffel matrix followed by implementing continuity of stress and normal displacement at the boundary); however, with the availability of highly efficient numerical linear algebra packages (such as BLAS/LAPACK) these equations are typically solved using a numeric eigenvalue / eigenvector analysis (e.g., Mallick and Frazer, 1990). Unfortunately, these exact solutions (both numerically and analytically derived) provide little to no insight into how certain parameters effect seismic reflectivity. For these reasons, estimating seismic reflectivity with approximate equations that are simpler and more intuitive is often desirable. This desire ultimately gave birth to several approximations that describe seismic reflection amplitude as a function of source-to-receiver offset, otherwise known as Amplitude-vs-Offset (AVO).

AVO equations started becoming popular in the early 1980’s. The Aki-Richards AVO approximation (Aki and Richards, 1980) and the Shuey AVO approximation (Shuey, 1985) are two isotropic AVO approximations from this time period that are still widely used today. These equations are expressed in terms of seismic incident angle, not offset, though these properties are related via Dix equations (Dix, 1955). As time passed AVO approximations became more complex. The Ruger approximation (Ruger and Tsvankin, 1997) expands AVO to Horizontally
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Transverse Isotropic (HTI) media, and includes an azimuthal component. AVO equations that describe amplitude as a function of angle/offset and azimuth are often referred to as AVAz.

The current trend in the geophysics community has been to develop AVO / AVAz approximations that algebraically include more and more terms (e.g., Downton et al., 2011 incorporates 4th order effects into an AVAz expression). There is inherently nothing wrong with this trend, and in fact these higher-order approximations often prove to be useful. To build on the previous example, incorporating 4th order effects into AVAz allows one to potentially resolve the fracture azimuth without a 90 degree ambiguity using minimal a priori information about the host rock and can help improve accuracy in fracture density estimation (Barone and Sen, 2018). However, these “improved” expressions frequently require much more a priori information and are often presented strictly as a way to improve modeling accuracy, with little to no discussion on how they help one to better estimate the underlying properties of interest (either via qualitative or quantitative interpretation). As such, one is often left with an accurate but complicated approximation that loses much of the predictive power that simpler AVO approximations have (e.g., via slope-intercept crossplotting). Additionally, the intuitive understanding that simpler AVO approximations often provide is lost (e.g., recognizing a Class-III AVO anomaly, or a “bright spot”, as a potential target). This simple intuition is often extremely useful in developing data classification schemes – while a quantitative approach for determining properties of interest (lithology, porosity, pore fluid, etc.) has obvious advantages, developing such an approach is far easier with an intuitive understanding of why it works. Ideally, one would want the “best of both worlds”: an AVO expression that is accurate enough to allow for quantitative interpretation and that includes the primary effects of the property of interest, but which also provides a simple and intuitive method for human-based interpretation. We believe that AVO approximations created using the Hybrid Born-Rytov (HBR) framework fit these criteria. Furthermore, our implementation of HBR AVO allows one to easily tailor the AVO approximation to any level of anisotropy without complicating the underlying equations. An accurate and understandable method/attribute is particularly beneficial in an anisotropic setting, since (excluding AVAz-based fracture density and azimuth estimation) there are few (perhaps none) anisotropic AVO attributes that even remotely rival the usefulness of isotropic-AVO-derived methods/attributes (such as intercept-gradient crossplotting).

AVO approximations are fundamentally designed to estimate seismic reflectivity, which is dependent on the physics of seismic wave propagation. Two commonly used and relatively simple approximations for describing seismic wave propagation are the Born approximation and the Rytov approximation. In their simplest forms, both of these approximations rely on describing both the seismic wave-field and the physical subsurface parameters in terms of a “background” value and a “perturbation.” These approximations and their relationship to the
seismic scattering problem are well described throughout the literature (e.g. Shaw and Sen (2004), Oristaglio (1985), Rajan and Frisk (1989), Hudson and Heritage (1981)). Some problems are more accurately described using the Born approximation while others benefit more from using the Rytov approximation; however, the Born approximation is especially prevalent in a wide range of geophysical methods and is used in the derivation of many common geophysics equations.

Marks (2006) showed that the Born and Rytov approximations can actually be described as end members of a larger class of “Hybrid” approximations. This Hybrid Born-Rytov (HBR) methodology includes a “tuning parameter” that transitions the approximation between one that is more Born-like to one that is more Rytov-like, and allows solutions to be modified based on the problem being solved. Marks (2006) describes this method in the context of optics, and as such his implementation only considers acoustic wave propagation. In this paper, we will:

1. Show how the HBR methodology can be extended to wave propagation in anisotropic elastic media by making analogy to the (linearized) Lippman-Schwinger Equation.
2. Describe a method for developing AVO approximations within the HBR framework by applying a tuning-parameter-dependent correction to a Born-based AVO expression.
3. Describe a proposed extension to the “standard” HBR framework coupled with a method for automatically determining the tuning parameter \( n \) that mathematically minimizes the difference between any \( R_{pp}^{HBR} \) and any target reflectivity curve \( R_{pp}^{True} \).
4. Present some initial work on the utility of using \( n \) as a tool for interpretation.

We believe HBR AVO approximations will simultaneously allow for more accurate modeling and lead to improved interpretation techniques, increasing the usefulness of AVO information.

This paper begins with a short summary of the paper, followed by an Introduction section describing the motivation and historical background behind this work. Next, background information and formulae relevant to this work will be provided. Subsequently, three novel methods for developing AVO approximations within the Hybrid Born-Rytov framework will be described in detail and compared using a standard stratigraphic model. One method is clearly superior to the others, both in terms of both theoretical and practical accuracy, though all three are shown for the sake of completeness. Lastly, this paper describes a method to automatically determine an ideal tuning parameter \( n \) and shows some preliminary work on the usefulness of \( n \) in the context of interpretation. This paper ends with a brief conclusion that overviews important findings and outlines possible future work. References and Appendices are provided at the end of the document.
BACKGROUND

In this section we will overview the fundamental theory behind the Hybrid Born-Rytov (HBR) method, and will show some of the solutions described by Marks (2006) for acoustic wave propagation. We will then relate these acoustic solutions to elastic solutions by comparing them to the (linearized) Lippmann-Schwinger equation. Next, we will overview the method presented in Shaw and Sen (2004) for developing AVO approximations in elastic media using a Born approximation. Finally, we will proposed an extension to the standard HBR framework coupled with a technique to automatically calculate the optimal $n$ for any pre-critical reflection.

The Original Hybrid Born-Rytov Method

This section will outline the theory underlying the HBR framework. The most important/relevant aspects will be overviewed here, though a more thorough description can be found in Appendix A as well as in Marks (2006), where the theory was originally described for use in Optics problems. This method combines two different approximations to wave propagation – the Born approximation and the Rytov approximation – into a single framework.

In the Born approximation, the wavefield ($u$) is assumed to be a combination of some reference wavefield ($u_0$) and some perturbation ($\Delta u$). Alternately, In the Rytov Approximation, the wavefield is a combination of some reference wavefield ($u_0$) and a phase term ($\phi$).

$$
\text{Born: } u_{\text{Born}} = u_0 + \Delta u \rightarrow u_{\text{Born}} = u_0 (1 + \phi), \quad \phi_{\text{Born}} = \frac{\Delta u}{u_0} = \frac{u - u_0}{u_0},
$$

$$
\text{Rytov: } u_{\text{Rytov}} = u_0 e^\phi \rightarrow \phi_{\text{Rytov}} = \ln \frac{u}{u_0}.
$$

These wavefield ($u$) and phase ($\phi$) approximations can be clearly related by rewriting them as:

$$
\text{Born: } u_{\text{Born}} = \lim_{n \to 1} \left(1 + \frac{\phi}{n}\right)^n \quad \text{and} \quad u_{\text{Rytov}} = \lim_{n \to \infty} \left(1 + \frac{\phi}{n}\right)^n,
$$

$$
\phi_{\text{Born}} = \lim_{n \to 1} n \left(\left(\frac{u}{u_0}\right)^{1/n} - 1\right) \quad \text{and} \quad \phi_{\text{Rytov}} = \lim_{n \to \infty} n \left(\left(\frac{u}{u_0}\right)^{1/n} - 1\right).
$$

By using this form and allowing $n$ to be a variable that ranges between 1 and $\infty$ rather than being described with a limit, we get the basic fundamental form of the HBR approximation:
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\[ u_{HBR} = u_0 \left( 1 + \frac{\phi}{n} \right)^n \quad \text{and} \quad \phi_{HBR} = n \left( \frac{u}{u_0} \right)^{1/n} - 1 \quad \text{where} \quad 1 \leq n \leq \infty . \]  

(5)

Clearly, at the lower limit of \( n = 1 \) the HBR approximation reduces to the Born approximation and in the upper limit of \( n = \infty \) the HBR approximation reduces to the Rytov approximation. The variable \( n \) is referred to as a “tuning parameter” that adjusts the solution between one that is more Born-like and one that is more Rytov-like. The ideal value of this parameter will vary problem to problem. Note that \( 2^35^1 \) is analogous to \( \Delta^2 \), and thus, in a general sense, the phase term \( \phi \) can be interpreted as some measure of “normalized difference” between the full wavefield \( u \) and reference wavefield \( u_0 \).

Relation to the Lippmann-Schwinger Equation

By combining \( u_{HBR} \) with an acoustic wave equation \((\nabla^2 u + k^2 u = 0)\), one can derive a solution (shown below) that is analogous to the Lippmann-Schwinger equation, which describes seismic scattering using the Born approximation. See Appendix A for the full derivation of Equation 6 and 7.

\[ \nabla^2 (u_0 \phi) + k_0^2 (u_0 \phi) = -u_0 \frac{n-1}{n} \left( 1 + \frac{\phi}{n} \right)^{-1} (\nabla \phi \cdot \nabla \phi) - \varepsilon \kappa u_0 \left( 1 + \frac{\phi}{n} \right) , \]  

(6)

\[ u_0 (r') \phi (r') = - \int_V d^3 r G (r, r') u_0 (r) \left[ \frac{n-1}{n} \left( 1 + \frac{\phi}{n} \right)^{-1} (\nabla \phi \cdot \nabla \phi) + \varepsilon \kappa (r) \left( 1 + \frac{\phi}{n} \right) \right] , \]  

(7)

where \( k \) is the wave-number, \( u_0 \) is the reference wavefield associated with \( k_0 , \varepsilon \) is an ordering parameter, \( r \) and \( r' \) represent position, and \( \kappa = k^2 - k_0^2 \). Reference parameters \( u_0 \) and \( k_0 \) are related such that \( \nabla^2 u_0 + k_0^2 u_0 = 0 \). Equations 6 – 7 reduce to the Lippmann-Schwinger equation in the Born-limit of \( n = 1 \). The 1st order approximation to Equation 7 is shown below:

\[ u_0 (r') \phi (r') = - \int_V d^3 r G (r, r') u_0 (r) \kappa (r) . \]  

(8)

For reference, the linearized Lippmann-Schwinger equation is shown below.

\[ u (r') - u_0 (r') = \Delta u (r') = - \int_V d^3 r G (r, r') u_0 (r) \kappa (r) . \]  

(9)

From comparing Equations 8 and 9, it is obvious that the 1st order Hybrid Born-Rytov solution and the linearized Lippmann-Schwinger equation are closely related. Specifically, the
“Right Hand Side” of the equations (i.e., the integral solution) is identical regardless of the value of $n$, though the “Left Hand Side” is not. Note that in the Born limit $u_0(r')\phi(r') = \Delta u(r')$, making Equations 8 and 9 identical. The close relationship between the 1st order Hybrid Born-Rytov solution and the linearized Lippmann-Schwinger equation is important because it indicates that existing AVO derivation methods that rely on the Born approximation can be modified for use with the Hybrid Born-Rytov method. In particular, these methods must be modified to account for the fact that, in general (i.e., for $n \neq 1$), $u_0(r')\phi(r') \neq \Delta u(r')$.

Note 1: Appendix A briefly describes the method presented in Marks (2006) for constructing higher order HBR solutions in acoustic media. These higher order solutions form a “Hybrid Series”, which can be related to a higher order Born series (see Marks, 2006).

Note 2: there have not yet been any constraints or assumptions regarding the elastic nature of the medium nor the anisotropy level, and is thus capable of being applied in anisotropic media. Everything thus far has been in terms of a generalized wavefield.

Using the Lippmann-Schwinger Equation to Develop Elastic HBR Solutions

This section will briefly outline how the Hybrid Born-Rytov solution in acoustic media described above can be transformed into an elastic solution. AVO approximations are designed to model elastic media, and thus a Hybrid Born-Rytov solution in elastic media is required prior to developing AVO solutions. Appendix B contains a short derivation of the Lippmann-Schwinger equation.

In general, the Lippmann-Schwinger equation takes the form:

$$\Delta u(x, \omega, x_s) = \int dx' g_0(x, \omega, x') \Delta L(x', \omega) u(x', \omega, x_s) ,$$

where

$$\Delta L(x, \omega) = L(x, \omega) - L_0(x, \omega) ,$$

$x$ is the wavefield location, $x_s$ is the source location, $\Delta u(x, \omega, x_s)$ and $u(x, \omega, x_s)$ are the scattered and the complete wavefield at frequency $\omega$ and location $x$ resulting from a source at $x_s$, respectively, and $L(x, \omega)$ and $L_0(x, \omega)$ are wavefield propagation operators for the complete and reference wavefield, respectively. The complete, reference, and scattered wavefields are related via the Born approximation shown in Equation 1. Refer to Appendix B for a derivation of Equation 10.
Because the Lippmann-Schwinger equation requires the complete wavefield \((u)\) to determine the scattered wavefield \((u_0)\), and \(u\) is unknown, directly solving for the scattered wavefield \((\Delta u)\) is impossible. As such, the linearized Lippmann-Schwinger Equation is typically used. This equation assumes that \(\frac{|u_0|}{|u|} \gg \frac{|\Delta u|}{|u|}\), and takes the form:

\[
\Delta u(x, \omega, x_s) = \int dx' g_0(x, \omega, x') \Delta L(x', \omega) u_0(x', \omega, x_s) .
\]  
(12)

The linearized Lippmann-Schwinger equation can be applied to different situations by altering the type of wavefield propagation operator used. When an acoustic operator (shown below) is used:

\[
L(x, \omega) = \left[ \nabla^2 - \frac{\omega^2}{c^2} \right] = \left[ \nabla^2 - k^2 \right] ,
\]

\(L_0(x, \omega) = \left[ \nabla^2 - \frac{\omega^2}{c_0^2} \right] = \left[ \nabla^2 - k_0^2 \right] ,
\]

the linearized Lippmann-Schwinger equation reduces to:

\[
\Delta u(x, \omega, x_s) = \int dx' g_0(x, \omega, x') \left( k^2 (x', \omega) - k_0^2 (x', \omega) \right) u_0(x', \omega, x_s) .
\]  
(15)

By making the substitution that \(\kappa = k^2 - k_0^2\), it is clear that the acoustic linearized Lippmann-Schwinger equation shown in Equation 15 and the 1st order Hybrid Born-Rytov solution shown in Equation 8 are identical in the Born-limit of \(n = 1\). Outside of the Born-limit, these equations only vary in the sense that \(\Delta u\) is replaced by \(u_0 \phi\).

Using an acoustic wave equation operator clearly illuminates the link between the Lippmann-Schwinger equation and the Hybrid Born-Rytov solution; however, the Earth is not acoustic. To account for this, \(L(x, \omega)\) can be defined as an elastic and anisotropic wave equation operator, shown below:

\[
L(x, \omega) = \rho \omega^2 + \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial}{\partial x_l} \right) ,
\]

\(L_0(x, \omega) = \rho_0 \omega^2 + \frac{\partial}{\partial x_j} \left( c_{ijkl}^0 \frac{\partial}{\partial x_l} \right) ,
\]

where \(\rho\) is density and \(c_{ijkl}\) is the elastic stiffness tensor (described using Einstein-index notation). Using these \(L(x, \omega)\) operators, the linearized Lippmann-Schwinger equation becomes:
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\[ \Delta u_n(x_r, \omega) = \int \psi d\psi' \left( \omega^2 \Delta \rho(x') u^0_i(x', \omega) G^0_{n1}(x', \omega; x_r) - \Delta c_{ijkl}(x') \frac{\partial u^0_x(x', \omega)}{\partial x_t} \frac{\partial c^0_{n1}(x', \omega; x_r)}{\partial x_j} \right), \]  \(18\)

where \(\Delta \rho\) and \(\Delta c_{ijkl}\) are described using the standard Born perturbation. Equation 18, often referred to as “the” Born equation, forms the basis of the Born AVO method that will be modified to produce Hybrid Born-Rytov AVO approximations.

**Deriving Born-based AVO Approximations**

This section will briefly outline how the linearized Lippmann-Schwinger equation can be used to develop AVO expressions in elastic, anisotropic media. This derivation closely follows the method described in Shaw and Sen (2004). Isotropic AVO solutions will be explicitly stated, though the path for developing analogous anisotropic AVO solutions will be clearly outlined. A few important equations and results will be included here, though a more complete derivation of these results is provided in Appendix C.

Shaw and Sen (2004) show that, based on a scattered wavefield of the form shown in Equation 18, the seismic reflection coefficient \(R_{pp}^{\text{Born}}\) is approximated as:

\[ R_{pp}^{\text{Born}}(\theta_i) = \frac{\Delta \rho [g_i g^r_i]_{r=r_0} + \Delta c_{ijkl} [g_i^r p^r_j g^r_k p^r_l]_{r=r_0}}{4 \rho_0 \cos^2 \theta_i}, \]  \(19\)

where \(\rho\) is density, \(c_{mn}\) are elastic coefficients, \(\theta_i\) is the seismic incident angle, \(\Delta\) indicates a standard Born perturbation, and \(g\) and \(p\) represent unit vectors describing the polarization and propagation of the incident (\(g\) and \(p\)) and scattered (\(g^r\) and \(p^r\)) wavefields. In an isotropic medium, \(g\) and \(p\) are described using Equation C19, and the stiffness tensor perturbation \(\Delta c_{ijkl}\) is described (in Voight notation) as:

\[ \Delta c_{ijkl} \rightarrow \Delta c_{mn} = \begin{bmatrix} \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & 0 & 0 & 0 \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & 0 & 0 & 0 \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta c_{55} \end{bmatrix}. \]  \(20\)

By substituting these variables into Equation 19 and summing over relevant indices, we find the following isotropic AVO approximation for \(R_{pp}\) to be:

\[ R_{pp}^{\text{Born}}(\theta_i) = \frac{\Delta \rho \cos(2\theta_i) + \frac{1}{\kappa^2_0} (\Delta c_{33} - 2\Delta c_{55} \sin^2(2\theta_i))}{4 \rho_0 \cos^2 \theta_i}, \]  \(21\)
where $\alpha$ is the P-wave velocity. Equation 21 is identical to the 3-term Aki-Richards AVO approximation (Aki and Richards, 1980; Shaw and Sen, 2004). In a more general anisotropic setting, the solutions produced by Equation 19 are consistent with the “weak anisotropy” reflectivity approximations (Psencik and Gajewski, 1998; Psencik and Vavrycuk, 1999; Psencik and Martins, 2001; and related papers). The following section describes how these Born-based solutions can be modified to be consistent with the HBR framework. We give explicit solutions for isotropic media (based off of Equation 21), though the methodology can be applied to any desired anisotropy level by simply replacing the Born reflectivity with the appropriate anisotropic version.

Note that effectively all commonly used AVAz can be derived by manipulating and making additional approximations to the aforementioned weak anisotropy reflectivity approximations – Barone and Sen (2018) show this for vertically fractured isotropic (VFI – effective HTI) as well as vertically fractured transverse isotropic (VFTI – effective orthorhombic) media. This paper does not rigorously investigate the potential benefit of using HBR for AZAz related workflows (such as those designed to estimate fracture properties), though from a theoretical viewpoint the following methodology can be applied to develop analogous HBR AVAz approximations.

**METHODOLOGY**

In this section, we will describe three methods for developing AVO approximations within the HBR framework. These methods all work by modifying a Born-based AVO approximation to incorporate aspects of the HBR structure. Explicit isotropic HBR AVO expressions based on Equation 21 will be included here, though we again reiterate that any anisotropic Born-based reflectivity approximation could be used instead.

We begin this section by quickly overviewing two methods that did not work very well, though nonetheless investigate an alternate path to determining HBR AVO solutions. These methods, which we generically label as Method 1 and Method 2, are based on the (questionable) implicit assumption that the form of the wavefield perturbation and the perturbation are related. Next, we derive and explain the HBR AVO method actually recommended/proposed by the authors. We typically refer to this as “The HBR AVO Method”, though to maintain a constant naming convention we will also refer to it as Method 3. Lastly, we describe a method to automatically determine $n$ to match any real (i.e., pre-critical) $R_{pp}^{\text{Born}}$ to any desired real $R_{pp}^{\text{True}}$. The ability to generate an arbitrary match between $R_{pp}^{\text{Born}}$ and $R_{pp}^{\text{True}}$ inherently requires $n_{\text{Best}}$ outside of the range $1 < n < \infty$, and thus this automatic method to determine $n_{\text{Best}}$ is inherently linked with a fundamental expansion to the HBR framework.
**Methods 1 and 2**

In the Born approximation, the wavefield perturbation is defined as $\Delta u = u - u_0$. This Born-style perturbation, however, is also present in the parameters in the linearized Lippmann-Schwinger equation (e.g., $\Delta \rho = \rho - \rho_0$). Thus, by modifying the perturbation parameters $\Delta \rho$ and $\Delta c_{ijkl}$ in the same way as $\Delta u$ is modified in the HBR method one can (perhaps) implicitly incorporate aspects of the HBR framework into the standard Born-perturbation equations. We refer to this process as “hybridizing” the variables. *Methods 1 and 2* both rely on this hybridization, but implement it in different ways. It is worth reiterating that these two methods are *not* recommended by the authors and are only included for the sake of completeness – the third method is considerably more favorable both from a theoretical and a practical viewpoint.

A given perturbation parameter $\Delta x$ can be redefined as

$$\Delta x_1 = x - x_0 .$$  (22)

Alternately, a given perturbation parameter $\Delta x$ in the Born method is directly represented in the Hybrid Born-Rytov method as $x_0\phi$. When $\phi$ is expanded to its complete form, this produces:

$$\Delta x_2 \rightarrow x_0\phi = x_0\left(n\left(\frac{x}{x_0}\right)^{1/n} - 1\right) .$$  (23)

Recall the basic HBR definitions shown in Equation 5. By assuming the generic variable $x$ has the same form, namely

$$x = x_0\left(1 + \frac{\phi}{n}\right)^n \quad \text{and} \quad \phi = n\left(\frac{x}{x_0}\right)^{1/n} - 1 ,$$  (24)

and by utilizing Equation 24 and by re-casting Equations 22 – 23 to be in terms of $\Delta x_{Born}$, the perturbations from *Method 1* and *Method 2* ($\Delta x_1$ and $\Delta x_2$, respectively) can be described as a function of a standard Born perturbation ($\Delta x_{Born}$) and $n$, resulting in:

$$\Delta x_1 = x - x_0 = x_0\left(1 + \frac{\Delta x}{nx_0}\right)^n - x_0 .$$  (25)

$$\Delta x_2 \rightarrow x_0\phi = x_0\left(n\left(1 + \frac{\Delta x}{nx_0}\right)^{1/n} - 1\right) .$$  (26)
Using Equations 25–26 as two possible templates results in two ways to modify $\Delta \rho$ and $\Delta c_{ijkl}$:

\[
\Delta \rho^1 \rightarrow \rho - \rho^0 \rightarrow \rho^0 \left(1 + \frac{1}{n} \frac{\Delta \rho}{\rho^0}\right)^n - \rho^0 ,
\]

\[
\Delta c_{mn}^1 \rightarrow c_{mn} - c_{mn}^0 \rightarrow c_{mn}^0 \left(1 + \frac{1}{n} \frac{\Delta c_{mn}}{c_{mn}^0}\right)^n - c_{mn}^0 .
\]

\[
\Delta \rho^2 \rightarrow \rho^0 \phi \rightarrow \rho^0 \left(n \left(1 + \frac{\Delta \rho}{\rho^0}\right)^{1/n} - 1\right) ,
\]

\[
\Delta c_{mn}^2 \rightarrow c_{mn}^0 \phi \rightarrow c_{mn}^0 \left(n \left(1 + \frac{\Delta c_{mn}}{c_{mn}^0}\right)^{1/n} - 1\right) .
\]

The parameter substitutions shown above do not affect the derivation of the isotropic Born AVO equation described in Appendix C. Thus, by substituting parameters in $R_{pp}^{\text{Born}}$ (see Equation 19) with those shown in Either Equations 27–28 or Equations 29–30, one can produce HBR AVO expressions that are dependent on $n$. The final reflectivity expressions from Methods 1 and 2 ($R_{pp}^1(\theta_i)$ and $R_{pp}^2(\theta_i)$, respectively) are shown below in Equations 31–32.

\[
R_{pp}^1(\theta_i) = \frac{(\rho^0 \left(1 + \frac{1}{n} \frac{\Delta \rho}{\rho^0}\right)^n - \rho^0) \cos(2 \theta_i) + \frac{1}{a_0} \left(c_{33}^0 \left(1 + \frac{1}{n} \frac{\Delta c_{33}}{c_{33}^0}\right)^n - c_{33}^0\right) - 2 \left(c_{SS}^0 \left(1 + \frac{1}{n} \frac{\Delta c_{SS}}{c_{SS}^0}\right)^n - c_{SS}^0\right) \sin^2(2 \theta_i)}{4 \rho_0 \cos^2 \theta_i} .
\]

\[
R_{pp}^2(\theta_i) = \frac{n \rho^0 \left(1 + \frac{1}{n} \frac{\Delta \rho}{\rho^0}\right)^{1/n} - 1 \right) \cos(2 \theta_i) + \frac{1}{a_0} \left(n c_{33}^0 \left(1 + \frac{1}{n} \frac{\Delta c_{33}}{c_{33}^0}\right)^{1/n} - 1\right) - 2n c_{SS}^0 \left(1 + \frac{1}{n} \frac{\Delta c_{SS}}{c_{SS}^0}\right)^{1/n} - 1 \right) \sin^2(2 \theta_i)}{4 \rho_0 \cos^2 \theta_i} .
\]

**The HBR AVO Method**

The 3rd method described in this work for developing HBR AVO approximations diverges substantially from the two methods described above. The derivation begins by examining the 1st order HBR solution (Equation 8), which is dependent on the linearized Lippmann-Schwinger equation and on the term $u_0 \phi$. Using the full phase definition, we find that $u_0 \phi$ is equivalent to:

\[
u_0 \phi = u_0 \left(n \left(\frac{u}{u_0}\right)^{1/n} - 1\right) .
\]
Equation 33 can be recast in terms of a Born-style wavefield perturbation (denoted \( \Delta u_{\text{born}} \)) by substituting \( u = u_0 + \Delta u_{\text{born}} \).

\[
    u_0 \phi = u_0 \left( n \left( \left( \frac{1+\Delta u_{\text{born}}}{u_0} \right)^{1/n} - 1 \right) \right) = \Delta u_{\text{born}} \left( n \left( \frac{1+\Delta u_{\text{born}}}{u_0} \right)^{1/n} - 1 \right) = \frac{\Delta u_{\text{born}}}{\alpha(u,u_0,n)} .
\] (34)

Equation 34 recasts \( u_0 \phi \) into a quantity that is directly proportional to \( \Delta u_{\text{born}} \) using a correction factor \( \alpha(u,u_0,n) \) that is dependent on \( \frac{\Delta u_{\text{born}}}{u_0} \) and \( n \). Fortunately, it is possible to estimate \( \frac{\Delta u_{\text{born}}}{u_0} \), since reflectivity is by definition defined as the ratio of the reflected wavefield amplitude to the incident wavefield amplitude. Under the Born approximation, the reflected wavefield is equivalent to \( \Delta u_{\text{born}} \) and the incident wavefield is equivalent to \( u_0 \). Thus,

\[
    R_{pp}^{\text{born}} \approx \frac{\Delta u_{\text{born}}}{u_0} \rightarrow \alpha(u,u_0,n) = \alpha(R_{pp}^{\text{born}},n) ,
\] (36)

where \( R_{pp}^{\text{born}} \) indicates some generalized Born-based AVO approximation (e.g., Equation 19). Making this substitution we find that

\[
    u_0 \phi = \Delta u_{\text{born}} \left( n \left( \frac{(1+R_{pp}^{\text{born}})^{1/n}}{R_{pp}^{\text{born}}} \right) \right) = \frac{\Delta u_{\text{born}}}{\alpha(R_{pp}^{\text{born}},n)} \rightarrow \alpha(R_{pp}^{\text{born}},n) = \left( \frac{R_{pp}^{\text{born}}}{n \left( (1+R_{pp}^{\text{born}})^{1/n} \right)} \right) ,
\] (37)

We now have a way to directly relate \( u_0 \phi \) to \( \Delta u_{\text{born}} \) using known quantities. Using the relationship between \( u_0 \phi \) and \( \Delta u_{\text{born}} \), we find that the first order HBR solution (Equation 8) can be re-written as:

\[
    u_0(r') \phi(r') \alpha(R_{pp}^{\text{born}},n) = \Delta u_{\text{born}}(r') = \alpha(R_{pp}^{\text{born}},n) \left( - \int d^3r G(r,r') \kappa(r) u_0(r) \right) .
\] (38)

Ignoring the \( \alpha(R_{pp}^{\text{born}},n) \) term in Equation 38 simply results in the standard Born AVO approximation \( R_{pp}^{\text{born}} \). Thus, by re-incorporating the \( \alpha(R_{pp}^{\text{born}},n) \) term, we find that the HBR AVO approximation \( R_{pp}^{\text{HBR}} \) can be expressed as:

\[
    R_{pp}^{\text{HBR}}(\theta_i,n) = \alpha(R_{pp}^{\text{Born}}(\theta_i),n) R_{pp}^{\text{Born}}(\theta_i) = \left( \frac{R_{pp}^{\text{Born}}(\theta_i)^2}{n \left( (1+R_{pp}^{\text{Born}}(\theta_i)^{1/n} \right)} \right) .
\] (39)
A Method to Automatically Determine the Tuning Parameter (n)

In the paper that originally outlined the HBR framework, Marks (2006) outlined a method for automatically determining what \( n \) should be. We refer you to the paper for the mathematical details, but the method was fundamentally based on the idea that one could analyze the dropped higher order terms in the HBR solution (refer to Equations 6 – 8) and choose a value of \( n \) that minimized the effect of some or all of these terms. Our method is based on this idea as well, but is implemented quite differently. In particular, we take advantage of the fact that we generally know what the “true” solution should be – in isotropic media this is described by the Zoeppritz equations, and in anisotropic media the analogous solution is described by the Christoffel system.

By phrasing the problem in terms of some desired “true” reflectivity, denoted \( R_{pp}^{\text{true}}(\theta_i) \), the ideal value of \( n \) can be found be solving the following minimization problem:

\[
 n_{\text{Best}} = \min_n \left| R_{pp}^{\text{true}}(\theta_i) - R_{pp}^{\text{HBR}}(\theta_i, n) \right| = \min_n \left| R_{pp}^{\text{true}}(\theta_i) - \frac{(R_{pp}^{\text{Born}}(\theta_i))^2}{n(1+R_{pp}^{\text{Born}}(\theta_i))^{1/n}-1} \right|, \quad (40)
\]

where \( n_{\text{Best}} \) is the ideal tuning parameter and \( n_{\text{Best}} = f\left( R_{pp}^{\text{true}}(\theta_i), R_{pp}^{\text{Born}}(\theta_i) \right) \). Assuming there exists a solution for \( n \) such that \( R_{pp}^{\text{true}}(\theta_i) = R_{pp}^{\text{HBR}}(\theta_i, n) \), Equation 40 can be re-written as:

\[
 n \left( \left( 1 + R_{pp}^{\text{Born}}(\theta_i) \right)^{1/n} - 1 \right) = \frac{(R_{pp}^{\text{Born}}(\theta_i))^2}{R_{pp}^{\text{true}}(\theta_i)}.
\]

(41)

It can be shown that the value of \( n \) needed to make Equation 41 valid is:

\[
 n_{\text{Best}} \left( R_{pp}^{\text{true}}, R_{pp}^{\text{Born}} \right) = \frac{-\log(1+R_{pp}^{\text{Born}}(\theta_i)}{\log(1+R_{pp}^{\text{Born}}(\theta_i)} + \mathcal{W}_m \left\{ \left( \frac{R_{pp}^{\text{true}}(\theta_i)}{(1+R_{pp}^{\text{Born}}(\theta_i))^{1/n}} \right)^2 \right\}, \quad (42)
\]

where \( \log \) is the natural logarithm and \( \mathcal{W}_m \{ (...) \} \) is the \( m \)th branch of the Lambert-W function, which provides the functional inverse to \( \text{we}^w \) (i.e., \( \mathcal{W}_m \{(w)\} = F^{-1}\{(\text{we}^w)\} \) → gives solution \( z \) that solves \( z = \text{we}^w \)). Note that by introducing the following three variable substitutions, Equation 42 can be described in a considerably more compact form:
Hybrid Born-Rytov AVO

\[ X = \left( \frac{R_{p}^{True}(\theta_i)}{R_{r}^{Born}(\theta_i)} \right)^2 \quad \text{and} \quad Y = \left( 1 + R_{p}^{Born}(\theta_i) \right)^{-X} \quad \text{and} \quad L = \log \left( 1 + R_{pp}^{Born}(\theta_i) \right), \quad (43) \]

\[ n_{Best} \left( R_{p}^{True}, R_{pp}^{Born} \right) = \frac{-L}{L X + \ln \left( -L X Y \right)} . \quad (44) \]

RESULTS AND DISCUSSION

Comparing the Three Proposed Methodologies

In this section we will compare the three Hybrid Born-Rytov AVO solutions described above using a simple isotropic stratigraphic model. Results for various values of \( n \) (i.e., the “Tuning Parameter”) will be shown. These results will be compared to the 2- and 3-term Aki-Richards AVO approximations as well as the full Zoeppritz solution. For reference, the Aki-Richards AVO approximation is provided below in Equation 45.

\[ R_{pp}^{AR} = \frac{1}{2} \frac{\Delta Z}{z_0} + \frac{1}{2} \left[ \frac{\Delta \alpha}{\alpha_0} - 4 \frac{\beta_0^2}{\alpha_0^2} \frac{\Delta \mu}{\mu_0} \right] \sin^2 \theta + \frac{1}{2} \frac{\Delta \alpha}{\alpha_0} \sin^2 \theta \tan^2 \theta , \quad (45) \]

where \( Z \) represents P-impedance, \( \alpha \) and \( \beta \) represents P- and S-wave velocity, and \( \mu \) represents the shear modulus. The “3-term” version is described exactly by Equation 48. The “2-term” version ignores the \( \frac{1}{2} \frac{\Delta \alpha}{\alpha_0} \sin^2 \theta \tan^2 \theta \) term, and is intended for use with smaller incident angles.

The stratigraphic model used in this work consists of (from top to bottom): Shale, Gas Saturated Sandstone, Brine Saturated Sandstone, and Shale. The specific layer properties of these layers can be seen in Figure 1. When calculating reflectivity curves it is necessary to define reference values and perturbation values for the layer properties. For any given property (e.g., density), the reference value was defined as the average between the upper and lower layer, whereas the perturbation was defined as the difference. These are defined for a generic property \( x \) as:

\[ \Delta x = x_2 - x_1 \quad \text{and} \quad x_0 = \frac{x_1 + x_2}{2} , \quad (46) \]

where \( x_1 \) and \( x_2 \) represent the lower and upper layer properties, respectively.
Figure 2 compares the three different HBR AVO approximations for the three layer interfaces shown in Figure 1. The blue-green shaded area represents a continuous range of \( n \) ranging from \( n = 1 \) (the Born-limit) in blue to \( n \approx 1.2 \text{ million} \) (the Rytov-limit) in green. In all images the red line represents the Zoeppritz solution and the dashed magenta and cyan lines represent the 2- and 3-term Aki-Richards AVO approximations, respectively. Note that the 3-term Aki-Richards AVO approximation is equivalent to the Born-limit HBR AVO approximation, despite being generated using a different formula (Equation 45). Within a given column, the axes on all sub-images are identical, allowing for a direct comparison between the three methods.

### MODEL

<table>
<thead>
<tr>
<th>Layer</th>
<th>( V_p )</th>
<th>( V_s )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shale</td>
<td>2770 m/s</td>
<td>1480 m/s</td>
<td>2.5 g/cc</td>
</tr>
<tr>
<td>Gas Sand</td>
<td>2790 m/s</td>
<td>1910 m/s</td>
<td>2.2 g/cc</td>
</tr>
<tr>
<td>Brine Sand</td>
<td>3200 m/s</td>
<td>1870 m/s</td>
<td>2.3 g/cc</td>
</tr>
</tbody>
</table>

Figure 1. The stratigraphic model used in this work to compare different isotropic AVO approximations. Three layer interfaces are present. Layers consist of (Top to Bottom): Shale, Gas Sand, Brine Sand, and Shale. Layer properties in terms of P-wave velocity \( (V_p) \), S-wave velocity \( (V_s) \), and Density \( (\rho) \) are shown in the figure.
Figure 2. A figure comparing the three Hybrid Born-Rytov (HBR) AVO solutions described in the text for each of the three layer interfaces in our synthetic model. The Rows represent the HBR AVO method used (top-to-bottom, these are represented by Equations 31, 32, and 40, respectively). The bottommost row shows results generated using the recommended HBR AVO method (i.e., “Method 3”). The columns represent the three interfaces in our synthetic model, and (left to right) represent a) Shale overlying a Gas-saturated Sandstone, b) Gas-saturated Sandstone overlying a Brine-saturated Sandstone, and c) Brine-saturated Sandstone overlying a Shale. Values of “tuning parameter” ($\eta$) ranging from 1 (Born-limit, shown in blue) to ~1.2 million (Rytov-limit, shown in green) are used. $\eta$ corresponding to the “sub-Born” ($0 < \eta < 1$) and “post-Rytov” ($\eta < 0$) regimes are not considered in these results. The Zoeppritz solution is shown in red, and the 2- and 3-term Aki-Richards AVO approximations are shown using the dashed magenta and cyan lines, respectively. Incident angles ($\theta$) ranging from 0 – 50 degrees are shown.
Hybrid Born-Rytov AVO

By comparing the results shown in Figures 2, one can make several interesting observations:

1. In all methods, the Born-limit solution is consistent, since in all 3 methods setting \( n = 1 \) results in no change being made relative to the Born solution. However, as \( n \) increases and \( R_{pp}^{HBR} \) becomes more Rytov-like, the three solutions diverge.

2. The Method 1 and Method 2 solutions are, in a way, opposites of each other. At a given incident angle (\( \theta \)), if the Method 1 solution at some \( n > 1 \) shows a higher (more positive) reflectivity relative to the Born-limit solution, then the Method 2 Rytov-limit solution always shows a lower (more negative) relative reflectivity estimate at that \( \theta \) and \( n \). The opposite is true as well. This can be explained by examining the perturbation replacements used by these methods (i.e., the “Hybridization” referred to in Equations 25 –26). We show this in Figure 3 for a general normalized perturbation \( \Delta x / x_0 \), which clearly indicates that (relative to the “standard” Born perturbation) Method 1 always increases perturbation \( (\Delta x / x_0) \) while Method 2 always decreases it, explaining this behavior. Unfortunately, neither of these solutions appears to be consistently correct. This may indicate that the implicit assumption underlying these methods, namely that a particular type of wavefield perturbation is inherently linked to the same type of parameter perturbation, is not valid.

3. The results from Methods 1 and 2 are largely inconsistent, although the results from the recommended HBR method (i.e., Method 3) consistently improve accuracy of \( R_{pp}^{HBR} \) relative to the Aki-Richards AVO approximations. This appears to remain true in many (though not all) other scenarios that were tested but that are not shown in this paper.

4. In the recommended HBR AVO method, the strength of the increase is directly related to the magnitude of \( R_{pp}^{Born} \). This “direct relationship” is seen in Equation 39, which shows that the correction factor applied to \( R_{pp}^{Born} \) is proportional to \( R_{pp}^{Born} \) (an additional dependence to \( R_{pp}^{Born} \) is present in the \( \alpha(R_{pp}^{Born}, n) \) term, producing the \( (R_{pp}^{Born}(\theta))^{2} \) term in the numerator). The requirement that increasing \( n \) results in \( R_{pp}^{HBR} \) becoming more positive can be verified by plotting \( \alpha(R_{pp}^{Born}, n) \) (i.e., the “correction factor” applied to \( R_{pp}^{Born} \) to compute \( R_{pp}^{HBR} \)) as a function of \( n \) and \( R_{pp}^{Born} \) (defined as \( \Delta u / u_0 \)). This crossplot is shown in Figure 4, which confirms that for all \( n > 1 \):

\[
R_{pp}^{Born} > 0 \implies \alpha(R_{pp}^{Born}, n) < 1 \implies R_{pp}^{HBR+} = \alpha(R_{pp}^{Born}, n)^{+} R_{pp}^{Born+} > R_{pp}^{Born+}, \tag{47}
\]

\[
R_{pp}^{Born} < 0 \implies \alpha(R_{pp}^{Born}, n) < 1 \implies R_{pp}^{HBR-} = \alpha(R_{pp}^{Born}, n)^{-} R_{pp}^{Born-} > R_{pp}^{Born-}. \tag{48}
\]
Figure 3. A comparison of how a given parameter perturbation ($\frac{\Delta x}{x_0}$) is substituted or “Hybridized” using “Method 1” and “Method 2”. Possible perturbations include density ($\rho$) and stiffness tensor elements $C_{33}$ and $C_{55}$. Perturbation values ranging from -0.5 to 0.5 are shown. Method 1 is shown in green, and Method 2 is shown in blue. For reference, the Born-limit solution (i.e., no change in the perturbation) is shown in red. It is clear that Method 1 always increases (makes more positive) the perturbation, whereas Method 2 always decreases (makes more negative) the perturbation.
Figure 4. A colormap showing how the correction factor $\alpha(R_{pp}^{Born}, n)$ used in the solution for $R_{pp}^{HBR}$ changes w.r.t. $n$ and $\frac{\Delta b}{u_0}$. The x-axis represents tuning parameter $n$ and ranges from 1 to ~60,000 on a logarithmic scale. The y-axis represents values of $\frac{\Delta b}{u_0}$, which is assumed to be identical to $R_{pp}^{Born}$, ranging from -0.3 to 0.3. The color represents the magnitude of the correction factor $\alpha(R_{pp}^{Born}, n)$, and ranges from 0.85 to 1.15. In this setup, $R_{pp}^{HBR} = \alpha(R_{pp}^{Born}, n) R_{pp}^{Born}$. Positive values of $\frac{\Delta b}{u_0}$ always result in $\alpha(R_{pp}^{Born}, n) > 1$, whereas negative values of $\frac{\Delta b}{u_0}$ always result in $\alpha(R_{pp}^{Born}, n) < 1$. The correction factor at the Born-limit of $n = 1$ is always 1, indicating that $R_{pp}^{HBR} = R_{pp}^{Born}$. See Equation 37 or 39 for the definition of $\alpha(R_{pp}^{Born}, n)$. 

Hybrid Born-Rytov AVO
Figure 4 confirms that “increasing $n$ always causes $R_{pp}^{HBR}$ to become more positive”, though does not adequately answer why this is the case. To answer why, one has to return to the fundamental HBR framework and the acoustic solutions presented by Marks (2006). Recall that the full Acoustic HBR solution is shown in Equations 6 – 7, but in order to make the solution not require a priori knowledge of the answer we used a 1st order approximation (shown in Equation 8). This 1st order solution is a function of $n$ on the left side (specifically, the phase term $\phi$), but is independent of $n$ on the other side. As such, as one varies $n$ the phase actually remains constant. This in turn implies that, in practice, the dependence on $n$ is in effect implemented by altering the wavefield such that the resultant phase is constant. Now, consider how the full wavefield estimate changes based on $\phi$. Using the equations for the complete wavefield $u$ shown in Equations 1 – 2, it is clear that for a given reference wavefield $u_0$ the “exact” born and Rytov solutions should produce the same wavefield $u$. Of course, these solutions require knowledge of the full wavefield to implement, making them untenable, but it shows what should happen in an idealized scenario. As such, by examining the phase required to maintain a consistent wavefield estimate at all $n$ and comparing it to the constant phase produced by the 1st order HBR solution, one can explain why increasing $n$ always increases reflectivity.

As a simple example, assume a scalar wavefield where $u = 1.1$ and $u_0 = 1$. Recall that $\phi_{Born} = \frac{u - u_0}{u_0}$ and $\phi_{Rytov} = \ln \frac{u}{u_0}$, meaning that for this example the “correct” phase is $\phi_{Born} = 0.1$ and $\phi_{Rytov} = \ln 1.1 \approx 0.09531$. If we replace $u$ with $u = 0.9$, we instead find that $\phi_{Born} = -0.1$ and $\phi_{Rytov} = \ln 0.9 \approx -0.10536$. This means that:

- **For $u > u_0$:** the Rytov phase should be less positive than the Born phase but isn’t, which in turn produces a larger complete wavefield, which in turn produces a larger difference between the reference and complete wavefields, which ultimately results in a more positive reflectivity.
- **For $u < u_0$:** the Rytov phase should be less negative than the Born phase but isn’t, which in turn produces a larger complete wavefield, which in turn produces a smaller difference between the reference and complete wavefields, which ultimately results in a more positive (or, more accurately, less negative) reflectivity.
The Major Limitation of the Original HBR Framework

The above analysis highlights what is a substantial limitation to the original HBR framework – namely, that it is only possible to produce \( R_{pp}^{HBR} \) in a very limited range for a given \( R_{pp}^{Born} \). Analysis of Equation 39 indicates that:

\[
\lim_{n \to \infty} R_{pp}^{HBR} (\theta_i, n) = R_{pp}^{Born} (\theta_i) = \lim_{n \to \infty} \frac{(r_{pp}^{Born}(\theta_i))^{2}}{n \log(1 + r_{pp}^{Born}(\theta_i))} = R_{pp}^{Born} (\theta_i) \left( \frac{r_{pp}^{Born}(\theta_i)}{\log(1 + r_{pp}^{Born}(\theta_i))} \right). \tag{49}
\]

In other words, if you define \( R_{pp}^{HBR} (\theta_i, n) = C(\theta_i, n) R_{pp}^{Born} (\theta_i) \) for some \( C(\theta_i, n) \), The original HBR framework can only produce solutions when \( C(\theta_i, n) \) is between 1 and \( \left( \frac{r_{pp}^{Born}(\theta_i)}{\log(1 + r_{pp}^{Born}(\theta_i))} \right) \). This is an extremely limited range, especially for the \( R_{pp}^{Born} \) magnitudes typically encountered in real-world usage. For example, for \( |R_{pp}^{Born}| = 0.1 \), the valid range of \( R_{pp}^{HBR} \) is roughly between \( R_{pp}^{Born} = 1 \) and 1.05 \( R_{pp}^{Born} \). In general terms, there are 3 regimes that are not accessible from within the standard HBR framework:

1. **The “sub-Born” regime**: The original HBR framework is not capable of producing solutions where \( R_{pp}^{HBR} < R_{pp}^{Born} \). These solutions can be accessed with \( 0 < n < 1 \).
2. **The “post-Rytov” regime**: The original HBR framework is not capable of producing solutions where \( R_{pp}^{HBR} > R_{pp}^{Born} \left( \frac{r_{pp}^{Born}}{\log(1 + r_{pp}^{Born})} \right) \). These solutions can be accessed with \( n < 0 \).
3. **The “mismatched sign” regime**: The original HBR framework is not capable of producing solutions where \( \text{sign}(R_{pp}^{HBR}) \neq \text{sign}(R_{pp}^{Born}) \). These solutions can be accessed by adding a small imaginary component to \( n \).

These additional regimes, in practice, allow any (real) \( R_{pp}^{Born} \) to be transformed into any (real) \( R_{pp}^{HBR} \). Fortunately, implementing these additional regimes does not require any additional work using our method – the formula for automatically estimating \( n \) (see Equations 42 – 44) will natively produce \( n \) with these extended characteristics when it is necessary to generate a match between the input reflectivity \( R_{pp}^{Born} \) and the target reflectivity \( R_{pp}^{HBR} \).

Figure 5 shows a combined color/contour-plot that describes \( R_{pp}^{HBR} \) as a function of \( n \) and \( R_{pp}^{Born} \). The red/blue coloring indicates the sign of \( R_{pp}^{Born} \), and the color magnitude indicates the magnitude of \( R_{pp}^{HBR} \). Contour lines indicate constant \( R_{pp}^{HBR} \) and are spaced at intervals of \( \delta R_{pp}^{HBR} = 0.1 \). It is useful to consider the tradeoff between \( n \) and \( R_{pp}^{Born} \) needed to maintain a constant \( R_{pp}^{HBR} \) value. Note: when considering solutions starting from “sub-Born” and going to “post-Rytov”, \( n \) will start at \( n = 0^+ \) (sub-Born) \( \rightarrow \) increase to \( n = 1 \) (Born-limit) \( \rightarrow \) increase to \( n = \infty \) (Rytov-limit) \( \rightarrow \) “wrap around” to \( n = -\infty \) \( \rightarrow \) increase to \( n = 0^- \) (post-Rytov).
Choosing the Correct Lambert-W Branch

There is one final piece of information needed to produce useful solutions for $n$ using Equations 42 – 44: the choice of which Lambert-W branch to use (defined by $m$). $\mathbf{W}_m\{(...)\}$, in general, will produce an infinite number of valid solutions to $F^{-1}\{we^w\}$ (where $F^{-1}\{(...)\}$ indicates a functional inverse operator). These solutions are commonly referred to as the different “branches” of the Lambert-W function. There is, however, one Lambert-W branch that is known as the “Primary Branch”. This branch is jointly defined by the $m = -1$ and $m = 0$ branches. Technically speaking, any Lambert-W branch could be used to produce a solution for $n$ that will enforce a match between $R_{pp}^{HBR}$ and $R_{pp}^{True}$, though it appears that only the primary Lambert-W branch has the potential to produce meaningful solutions for $n$ that align with other major features in the HBR framework. Figure 6 shows a plot of the primary Lambert-W branch and identifies segments represented by the $m = -1$ and $m = 0$ branches. Note that for the Lambert-W operation used here (i.e., $\mathbf{W}_m\{-LXY\}$), the Lambert-W operand ($-LXY$) is always in the range $-\frac{1}{e} < (-LXY) \leq 0$. This is trivial to show numerically by defining a grid of $R_{pp}^{Born}$ and $R_{pp}^{True}$ where each is allowed to take any value between $-1 < R_{pp} \leq 1$, and then computing the operand ($-LXY$) for each possible combination of $R_{pp}^{Born}$ and $R_{pp}^{True}$. The dependence on $\frac{1}{e}$ is explained by considering the $\mathbf{W}_0\{w\}$ series expansion around $w = 0$, which is shown below in Equation 50.
In the coefficient term \( \binom{k-1}{k!} \) in Equation 49, the \( k! \) denominator term increases faster than the \( k^{k-1} \) numerator term. If we consider how this tradeoff evolves as \( k \) increases, we find

\[
\lim_{k \to \infty} \frac{\binom{k+1}{k+1}}{\binom{k-1}{k!}} = \lim_{k \to \infty} \frac{k!}{(k+1)!} \left( \frac{k+1}{k} \right)^k = \lim_{k \to \infty} \frac{1}{k+1} \left( 1 + \frac{1}{k} \right)^k = e. \tag{51}
\]

Thus, in order to ensure convergence in the series of \( \mathcal{W}_0\{(w)\} \) with a coefficient that increases in magnitude by (up to) a factor of \( e \) every time the series order \( (k) \) is increased, the only stable \( w \) must be in the range \(-\frac{1}{e} < w \leq \frac{1}{e}\). The positive \( w \) bound is less strict because the series gets the benefit of cancelling terms thanks to the \((-1)^{k-1}\) term in the series. The above analysis, combined with Figure 6, which shows the \( \mathcal{W}_0\{(w)\} \) and \( \mathcal{W}_{-1}\{(w)\} \) branches connecting at a value of \( w = -\frac{1}{e} \), suggests that the boundary between \( \mathcal{W}_0\{(w)\} \) and \( \mathcal{W}_{-1}\{(w)\} \) in \([R_{pp}^{true}, R_{pp}^{Born}]\)-space may be related to the Rylov-Limit HBR solution. It turns out that this boundary is not just related to the Rylov-Limit HBR solution...it is defined by it. This is again trivial to check numerically, since (at a particular point in \([R_{pp}^{true}, R_{pp}^{Born}]\)-space) one of the Lambert-W branches will produce an obviously wrong answer while the other will produce something that is reasonable. Mathematically, the Lambert-W Branch selection criteria can be described as shown below in Equation 52. We do not show a step-by-step derivation since this is fairly easy to derive by either by considering Equation 41 in the limit of \( n \to \infty \) or by rearranging terms in Equation 49.

\[
\mathcal{W}_m: \quad m = \begin{cases} 0 & \text{if } \text{sign}\{(R_{pp}^{Born})\} = \text{sign}\{(R_{pp}^{true})\} \& \& |R_{pp}^{Born}| > \frac{\left(R_{pp}^{Born}(\theta_l)\right)^2}{e^{1 - R_{pp}^{true}(\theta_l)} - 1} \\ -1 & \text{Otherwise} \end{cases}. \tag{52}
\]

It is additionally worth noting that the substitution variables \( X, Y, \) and \( L \) (defined in Equation 43) are not assigned these values randomly. Rather, each of these quantities are present in Equation 39 in the limit of \( n \to \infty \). Note that \( \lim_{n \to \infty} \left(1 + x\right)^{1/n} - 1 = \log(1 + x) \).
In Figure 6 we plot a simple line graph showing the primary LambertW branch and that identifies what areas of the primary branch correspond to the $W_0$ and $W_{-1}$ branches. In Figure 7, we show the real and imaginary parts of $n$ as well as the LambertW branch required to match every possible real $R_{pp}^{\text{Born}}$ to every possible real $R_{pp}^{\text{True}}$. The match between $R_{pp}^{\text{HBR}}$ and $R_{pp}^{\text{True}}$ is near perfect – on a difference plot (not shown) the only non-zero mismatch is near areas where $n = \pm \infty$, and in these the mismatch is on the order of $\delta R_{pp} = 10^{-10}$, suggesting the only error is introduced from minor numerical round-off errors at $|n| \gg 1$.

![The Primary LambertW Branch](image)

**Figure 6.** A line-plot showing the primary Lambert-W Branch, which is jointly composed of the $W_0\{(w)\}$ and $W_{-1}\{(w)\}$ Lambert-W branches (shown in green and blue, respectively). All LambertW inputs $(w)$ relevant to the HBR problem are in the range $-\frac{1}{e} < w \leq 0$. 
Figure 7. A set of colorplots describing the $R_{pp}^{HBR}$ solution (and associated $n$) for all $R_{pp}^{Born}$ and $R_{pp}^{True}$. (Upper) Images showing the $R_{pp}^{HBR}$ solution for every possible combination of $R_{pp}^{Born}$ and $R_{pp}^{True}$. (Lower) Images showing the real and imaginary components of $n$ as well as the Lambert-W branch used to produce the $R_{pp}^{HBR}$ solution in the upper row. These results clearly shows that our method can modify any $R_{pp}^{Born}$ to generate a $R_{pp}^{HBR}$ solution that perfectly matches $R_{pp}^{True}$. All $R_{pp}$ are real: we are not currently considering imaginary (post-critical) $R_{pp}$. Note that $n_{imag}$ only exists when $\text{sign}(R_{pp}^{Born}) \neq \text{sign}(R_{pp}^{True})$, and in these cases $n_{imag}$ is typically several orders of magnitude smaller than $n_{Real}$.

Initial Results Regarding The Tuning Parameter’s Potential for Subsurface Classification

The primary goal of this work is, in a general sense, to aid in extracting information about the subsurface from data, and to this end we believe that analyzing the tuning parameter ($n$) may prove to be a useful tool in both qualitative and quantitative classification schemes. This is an area we are actively working on and the results shown here are subject to change. However, while we cannot yet propose tuning-parameter-based classification guidelines, these initial results do seem to indicate that $n$ has the potential to aid in subsurface classification in at least some scenarios and is worth investigating further. It additionally suggests a few specific ways that $n$ could be quickly and easily used as an indicator for a few specific subsurface features and attributes.
Hybrid Born-Rytov AVO

Our initial work is performed on synthetically generated seismic data. This synthetic data is based off of a real well log sequence with 3 shale facies and 3 sandstone facies identified. Each sandstone facies is transformed into 2 sub facies – a brine saturated facies and an oil saturated facies. The overall sequence is then repeated 3 times and concatenated vertically.

- The 1st repetition uses finely interbedded oil- and brine-saturated sandstone packages everywhere a sandstone is present in the facies model.
- The 2nd repetition uses only brine-saturated sandstone facies.
- The 3rd repetition uses only oil-saturated sandstone facies.

This facies model is shown in Figure 8. Figure 9 shows the synthetic data generated using this model. Synthetics were generated using convolution-based modeling and a variable-peak-frequency Ricker wavelet. Lastly, Figure 10 shows the real component of the automatically calculated tuning parameter ($\eta$) as a function of depth and seismic incident angle. The color indicates the value of $\eta$, with cool colors indicating [sub]-Born, and warm colors indicating [post]-Rytov. The jagged vertical black lines show the “wrap-around angle” where $n = 0 \rightarrow n = -\infty$. All information within $\frac{1}{2}$ wavelength of a post-critical reflection was removed, resulting in 2 small areas of missing data at high (> $40^o$) incident angles.

A careful visual analysis of Figure 10 suggests multiple ways that $\eta$ might be related to specific facies, fluid types, and other subsurface properties of interest. More data is needed to say anything conclusive in this regard, though we will quickly outline two rather noticeable features that could be easily utilized in classification/interpretation.

1. The “background trend” seems to be Born-like HBR AVO at normal-incidence and a gradual increase up to post-Rytov HBR AVO as the seismic incidence angle ($\theta_i$) increases.

2. The finely interbedded sandstone package has an anamalous $n(depth, \theta_i)$ signal. In particular, the signal is characterized by an immediate reduction into the sub-Born regime at low incident angles, followed by a rapid transition to the post-Rytov regime with a below-average wrap-around angle.

3. The middle- and lower-sequence packages, corresponding to brine-saturated sands and oil-saturated sands, respectively, have a similar overall character. However, most features in $n(depth, \theta_i)$ appear to be shifted $5^o - 15^o$, with features occurring at higher incident angles in the oil-saturated case than in the brine-saturated case. Considering the inherent difficulty in distinguishing between two (non-gaseous) pore fluid types, this could be an especially useful attribute should this hold in more general scenarios.
Figure 8. An image showing what facies type is present at every depth sample for the model we used to generate synthetic data. The same facies package repeats 3 times with different pore-fluid in the sandstone facies. The boundaries between these repetitions are denoted by the two horizontal black lines. The general type (brine-saturated sandstone, oil-saturated sandstone, or shale), is identified using the purple/cyan/yellow background colors.
Figure 9. Plots of the synthetic seismic data generated using convolutional modeling and a reflectivity series defined using the 3-term Aki-Richards AVO approximation (Upper, corresponds to $R_{pp}^{\text{Born}}$) and Zoeppritz equations (Lower, corresponds to $R_{pp}^{\text{True}}$). Incident angle (0-50 degrees), depth range (1-2450 samples), and color range (-0.2 to 0.2) are identical in both images. Post-critical events have been removed.
Figure 10. A colorplot showing the automatically determined tuning parameter (π) as a function of depth and incident angle (θ). Angles range from 0-50 degrees. The 3 main sequence packages are separated by the two long horizontal black lines. The “finely interbedded sandstone sequence” discussed in the main text is identified (this is centered roughly on depth=1000). Post-Critical data from two areas at θ > 40° have been removed. Colors indicate the following regimes: Blue → sub-Born, White → Born, Yellow/Orange → Born-Rytov, Dark Orange → Rytov, Black → “wrap around angle”, Red/Purple → post-Rytov, Green → Zero (below color-scale precision). Cyan/Orange in sequences of Cyan/Green/Orange indicate ±1 colorscale sample from zero → smallest non-zero sub-Born/post-Rytov value that can be plotted.
CONCLUSIONS

Over the past several decades, AVO information has proven itself to be a useful tool when analyzing seismic data using both qualitative and quantitative approaches. Approaches such as “intercept-gradient crossplotting” and “AVO classes” (Rutherford and Williams, 1989) have proven extremely valuable in hydrocarbon exploration, in large part due to their simple and easy-to-understand implementation. Unfortunately, as soon as one moves out of the isotropic regime and begins to consider anisotropic media, nearly all of these simple and intuitive methodologies cease to exist. In fact, outside of a few select sub-areas such as AVAz-based fracture characterization, there does not seem to be any well-agreed upon framework for easily using/interpreting anisotropic AVO information. Furthermore, many recent AVO-related articles seem more focused on increasing modeling accuracy (often at the cost of needing additional a priori information) with little to no insight on how the proposed AVO method actually increases the usability of AVO. We believe that Hybrid Born-Rytov (HBR) AVO offers at least a partial solution to this dilemma, and incorporates both increased accuracy as well as the potential for an easy-to-interpret attribute that can be easily computed at any anisotropy level ranging from isotropic to triclinic. In this paper, we:

1. Overviewed the HBR framework as it was originally presented in Acoustics and described a way to “transform” the acoustic HBR solutions to elastic/anisotropic HBR solutions using the Lippman-Schwinger Equation.

2. Presented three methods (though only one is recommended) for developing elastic/anisotropic BHR AVO approximations by applying a “correction factor” to one of the (extremely commonplace) Born-based AVO expressions that is a function of $R_{pp}^{Born}$ and the HBR tuning parameter $n$.

3. Presented a methodology to automatically estimate the tuning parameter such that it will transform any $R_{pp}^{Born} (\theta_i)$ curve into any desired/target (real/pre-critical) reflectivity curve $R_{pp}^{True} (\theta_i)$. This method inherently requires an expansion to the base HBR theory, resulting in 3 new HBR regimes:
   - *Sub-Born*: defined by $R_{pp}^{True} (\theta_i) < R_{pp}^{Born} (\theta_i)$, accessed with $0 < n < 1$.
   - *Post-Rytov*: defined by $R_{pp}^{True} (\theta_i) > R_{pp}^{Rytov} (\theta_i)$, accessed with $-\infty < n < 0$.
   - *Mismatched Sign*: defined by $\text{sign}\{R_{pp}^{True} (\theta_i)\} \neq \text{sign}\{R_{pp}^{Born} (\theta_i)\}$, accessed with $\text{imag}\{n\} \neq 0$.

4. Presented some preliminary results using synthetic data to test the potential utility of using $n$ as an indicator of subsurface properties. More work needs to be done prior to drawing any definite conclusions, but in our test example the $n(\theta, \text{depth})$ plot included many features that correlated well with the model used to generate the data.
The steps described in this paper provide a strong theoretical background for the elastic/anisotropic HBR AVO framework, and describe straightforward method to develop HBR ABO solutions for any elastic/anisotropic medium by modifying a “standard” Born-based AVO expression. The main benefit of developing solutions in this manner is its practical usability – all the anisotropic effects are included within the $R_{pp}^{\text{Born}}(\theta)$ expression, which is well-known easy to compute for any anisotropy level using the weak-anisotropy equations (see Psencik and Gajewski, 1998; Psencik and Vavrycuk, 1999; Psencik and Martins, 2001; and related papers). The implication of this is that there is a single simple HBR AVO formula that is valid for all anisotropy levels between isotropic and triclinic, which allows anisotropic cases to be treated identically to isotropic ones. Our automatic method for choosing $n$ is based around practical usability as well – by simply inputting a desired target reflectivity curve (which can be numerically computed using a method that is more accurate but less intuitive), one is able to distil the problem down to a single attribute ($n$) using a single set of equations that are the same regardless of anisotropy level. As such, even though our research into using $n$ to aid in subsurface interpretation is still a work in progress, we expect that any $n$-dependent relationships we might uncover will also hold true regardless of anisotropy level. Ultimately, this gives HBR AVO the potential to provide one of the first simple and intuitive general-use classification/interpretation aides for use in arbitrary anisotropic media.
REFERENCES


APPENDIX A

Derivation of the Hybrid Born-Rytov Method in Acoustic Media

This Appendix closely follows the derivation presented in Marks (2006) that outlines the theoretical framework for the Hybrid Born-Rytov method in acoustic media.

The Hybrid Born-Rytov method comes from combining two different approximations to wave propagation – the Born approximation and the Rytov approximation. In the born approximation, a variable \( u \) is assumed to be a combination of some reference value \( u_0 \) and some perturbation \( \Delta u \), as shown in Equation A1.

\[
u = u_0 + \Delta u = u_0 \left(1 + \frac{\Delta u}{u_0}\right)
\]  
(A1)

Typically, it is assumed the perturbation is relatively small, such that

\[
|\frac{\Delta u}{u_0}| \ll 1
\]  
(A2)

In a more general sense, variable can be approximated using a Born series of the form

\[
u = \sum_{n=0}^{\infty} \varepsilon^n u_n = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots
\]  
(A3)

where \( \varepsilon \) is an ordering parameter. Alternately, in the Rytov approximation, a variable is a combination of some reference value \( u \) and a phase term \( \phi \):

\[
u = u_0 e^\phi \rightarrow \phi = \ln \frac{u}{u_0}
\]  
(A4)

\[
\phi = \ln \frac{u}{u_0} = \ln \frac{u_0 + \Delta u}{u_0} = \ln \left(1 + \frac{\Delta u}{u_0}\right) \approx \frac{\Delta u}{u_0}
\]  
(A5)

In a more general sense, variable can be approximated using a Rytov series, such that

\[
u = \exp \left(\sum_{n=0}^{\infty} \varepsilon^n \phi_n\right) = \exp \left(\phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots\right)
\]  
(A6)

By altering the exponential in the Rytov approximation to its limit form and removing the limit, we can see the basic form of the Hybrid Born-Rytov approximation.
Hybrid Born-Rytov AVO

\[ u = u_0 e^{\phi} = u_0 \lim_{n \to \infty} \left(1 + \frac{\phi}{n}\right)^n \Rightarrow u = u_0 \left(1 + \frac{\phi}{n}\right)^n, \quad (A7) \]

\[ 1 < n < \infty, \quad (A8) \]

where \( n \) is a “tuning parameter” which adjust the solution between one that is more born-like and one that is more Rytov-like. From Equations A7 and A8 we can see that at the lower limit of \( n = 1 \) the approximation reduces to the Born approximation, and in the upper limit of \( n = \infty \) the approximation reduces to the Rytov approximation. The ideal value of this parameter will vary depending on the individual problem being solved. The phase term in the hybrid method can be calculated by performing a similar analysis to the Rytov approximation phase term, as is seen in Equation A9.

\[ u = u_0 e^{\phi} \Rightarrow \phi = \ln \frac{u}{u_0} = \lim_{n \to \infty} n \left(\frac{u}{u_0}\right)^{1/n} - 1 \Rightarrow \phi = n \left(\frac{u}{u_0}\right)^{1/n} - 1. \quad (A9) \]

Again we can see that at the lower limit of \( n = 1 \) the hybrid approximation phase reduces to the Born approximation, and in the upper limit of \( n = \infty \) the hybrid approximation phase reduces to the Rytov approximation.

Next, we combine the Hybrid Born-Rytov method form shown in Equation A7 with an acoustic wave equation shown below (in Equation A10) and, after performing some mathematical analysis, we are able to derive an analogue to the linearized Lippmann-Schwinger equation.

\[ \nabla^2 u + k^2 u = 0, \quad (A10) \]

where \( u \) is the wavefield and \( k \) is the wavenumber / spatial frequency. Combining Equation A10 with Equation A7 produces

\[ \nabla^2 \left[u_0 \left(1 + \frac{\phi}{n}\right)^n\right] + k^2 \left[u_0 \left(1 + \frac{\phi}{n}\right)^n\right] = 0. \quad (A11) \]

Note that the reference field \( (u_0) \) is the solution to \( \nabla^2 u_0 + k_0^2 u = 0 \). Expanding and rearranging Equation A11 produces
Hybrid Born-Rytov AVO

\[
\left(1 + \frac{\phi}{n}\right)^n \nabla^2 u_0 + 2 \left(1 + \frac{\phi}{n}\right)^{n-1} \nabla u_0 \cdot \nabla \phi + u_0 \left(1 + \frac{\phi}{n}\right)^{n-1} \nabla^2 \phi + u_0 \frac{n-1}{n} \left(1 + \frac{\phi}{n}\right)^{n-2} (\nabla \phi \cdot \nabla \phi) + k^2 u_0 \left(1 + \frac{\phi}{n}\right)^n = 0
\]  

\text{(A12)}

Dividing by \( \left(1 + \frac{\phi}{n}\right)^{n-1} \) and substituting \( \kappa = k^2 - k_0^2 \) results in

\[
2 \nabla u_0 \cdot \nabla \phi + u_0 \nabla^2 \phi + u_0 \frac{n-1}{n} \left(1 + \frac{\phi}{n}\right)^{-1} (\nabla \phi \cdot \nabla \phi) + \varepsilon \kappa u_0 \left(1 + \frac{\phi}{n}\right) = 0
\]  

\text{(A13)}

In Equation A13 \( \varepsilon \) has been inserted to indicate that \( \kappa \) is of 1st order. By noting that

\[
\nabla^2 (u_0 \phi) = u_0 \nabla^2 \phi + 2 \nabla u_0 \cdot \nabla \phi + \phi \nabla^2 u_0,
\]  

\text{(A14)}

\[
\nabla^2 u_0 = -k_0^2 u_0,
\]  

\text{(A15)}

Equation A13 can be re-written as

\[
\nabla^2 (u_0 \phi) + k_0^2 (u_0 \phi) = u_0 \nabla^2 \phi + 2 \nabla u_0 \cdot \nabla \phi.
\]  

\text{(A16)}

Finally, by combining Equations A16 and A14, we arrive at the linearized Lippmann-Schwinger analogue equation:

\[
\nabla^2 (u_0 \phi) + k_0^2 (u_0 \phi) = -u_0 \frac{n-1}{n} \left(1 + \frac{\phi}{n}\right)^{-1} (\nabla \phi \cdot \nabla \phi) - \varepsilon \kappa u_0 \left(1 + \frac{\phi}{n}\right).
\]  

\text{(A17)}

Equation A17 reduces to the linearized Lippmann-Schwinger equation in the Born limit of \( n = 1 \). In general, Equation A17 has the solution:

\[
u_0(r') \phi(r') = -\int d^3 r G(r', r) u_0(r) \left[\frac{n-1}{n} \left(1 + \frac{\phi}{n}\right)^{-1} (\nabla \phi \cdot \nabla \phi) + \varepsilon \kappa(r) \left(1 + \frac{\phi}{n}\right)\right],
\]  

\text{(A18)}

where \( G(r', r) \) is the Green’s Function of the homogeneous wave equation. By grouping terms in Equation A18 by order of \( \phi \) and \( \varepsilon \) and dropping higher order terms, it can be shown that the 1st order approximate solution is
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\[ u_0 (r') \phi (r') = - \int_V d^3 r G (r, r') u_0 (r) \kappa (r) \]  \quad (A19)

Equation A19 is nearly identical to the linearized Lipmann Schwinger equation. For reference, the linearized Lipmann-Schwinger equation is shown below in Equation A20.

\[ u (r') - u_0 (r') = \Delta u (r') = - \int_V d^3 r G (r, r') u_0 (r) \kappa (r) \]  \quad (A20)

The integral on the right hand side is equivalent to the linearized Lipmann-Schwinger equation; however, left hand side is only identical in the Born limit of \( n = 1 \). The Lipmann-Schwinger equation is further described in Appendix B. Note that in the Born limit: \( u_0 \phi = u - u_0 = \Delta u \).

Marks (2006) showed that this Hybrid series could be further analyzed in order to group the solution into terms of different orders. Equation A13, when multiplied by a constant of \( \frac{1 + \phi / n}{u_0} \), becomes

\[
2 \frac{\nabla u_0}{u_0} \cdot \nabla \phi \left( 1 + \frac{\phi}{n} \right) + \nabla^2 \phi \left( 1 + \frac{\phi}{n} \right) + \frac{n-1}{n} (\nabla \phi \cdot \nabla \phi) + \epsilon \kappa \left( 1 + \frac{\phi}{n} \right)^2 = 0 . \quad (A21)
\]

By expanding Equation A21 and grouping terms based on order coefficient \( \epsilon^p \), it can be shown that

\[
2 \frac{\nabla u_0}{u_0} \cdot \nabla \phi_p + \frac{1}{n} \sum_{m=0}^{p-1} \phi_{p-m} \nabla \phi_m + \left( \nabla^2 \phi_p + \frac{1}{n} \sum_{m=0}^{p-1} \phi_{p-m} \nabla^2 \phi_m \right) + \frac{n-1}{n} \left( \sum_{m=1}^{p-1} \nabla \phi_{p-m} \cdot \nabla \phi_m + \kappa \left( \frac{2 \phi_{p-1}}{n} + \frac{n}{2} \sum_{m=0}^{p-1} \phi_{p-m-1} \phi_m \right) = 0 . \quad (A22)
\]

Rearranging terms in Equation A22 produces

\[
(\nabla^2 + k_0^2) (u_0 \phi_p) = - \left[ \frac{1}{n} \sum_{m=0}^{p-1} \phi_{p-m} ((\nabla^2 + k_0^2) (u_0 \phi_m)) + \frac{n-1}{n} u_0 \sum_{m=1}^{p-1} \nabla \phi_{p-m} \cdot \nabla \phi_m + \frac{\kappa}{n^2} u_0 \sum_{m=0}^{p-1} \phi_{p-m-1} \phi_m + \frac{2\kappa}{n} u_0 \phi_{p-1} \right] . \quad (A23)
\]

Equation A23 produces equations describing \((\nabla^2 + k_0^2)(u_0 \phi_p)\) for various orders of \( p \). The first 3 orders of \( p \) as well as a generalized solution to Equation A22 are shown below.
Hybrid Born-Rytov AVO

\[(\nabla^2 + k_0^2)(u_0 \phi_1) = -u_0 \kappa ,\]

\[(\nabla^2 + k_0^2)(u_0 \phi_2) = -u_0 \left[\frac{n-1}{n} (\nabla \phi_1 \cdot \nabla \phi_1) + \frac{1}{n} \phi_1 \kappa \right] ,\]

\[(\nabla^2 + k_0^2)(u_0 \phi_3) = -u_0 \left[\frac{n-1}{n} \left( 2\nabla \phi_1 \cdot \nabla \phi_2 - \frac{1}{n} \phi_2 (\nabla \phi_1 \cdot \nabla \phi_1) \right) + \frac{1}{n} \phi_2 \kappa \right] . \quad (A24)\]

In general, the solution to the Equations shown in A24 have the following solution:

\[(\nabla^2 + k_0^2)(u_0 \phi_p) = X(n, \phi_p, \kappa, u_o) \rightarrow (u_0 \phi_p) = \int_{-\infty}^{\infty} d^3r G(r', r)X(n, \phi, \kappa, u_o) , \quad (A25)\]

where \(X(n, \phi, \kappa, u_o)\) is some function of \(n, \phi, \kappa,\) and \(u_o.\)

Note that Equation A19 is the integral form of the 1\(^{\text{st}}\) order solution shown in Equation A24, and is related to the linearized Lippmann-Schwinger equation in the same way. The solutions shown in Equation A24 can largely be separated into Born and Rytov parts based on weighting terms — terms weighted by \(\frac{1}{n}\) represent a Born contribution (weight goes to one for \(n = 1\)), whereas terms weighted by \(\frac{(n-1)}{n}\) represent a Rytov contribution (weight goes to one for \(n = \infty\)).
APPENDIX B

Derivation of the Lippmann-Schwinger Equation

This Appendix will briefly outline how the Lippmann-Schwinger equation can be derived. The Lippmann-Schwinger is designed to predict seismic scattering, and uses a Born-like approach to derive. Initially, a wave equation operator is defined, such that

\[ L(x, \omega)u(x, \omega, x_s) = -a(\omega)\delta(x - x_s) \, , \]  

(B1)

where \( x \) is a location, \( x_s \) is the source location, \( a(\omega) \) is the source amplitude (in frequency domain), \( \delta \) is the dirac-delta function, \( u(x, \omega, x_s) \) is the wavefield at frequency \( \omega \) and location \( x \) resulting from a source at \( x_s \), and \( L(x, \omega) \) is some wavefield operator. Both \( L(x, \omega) \) and \( u(x, \omega, x_s) \) are then assumed to be comprised of some reference value and some perturbation, such that:

\[ L(x, \omega) = L_0(x, \omega) + \Delta L(x, \omega) \, , \]  

(B2)

\[ u(x, \omega, x_s) = u_0(x, \omega, x_s) + \Delta u(x, \omega, x_s) \, . \]  

(B3)

A Green’s function for the reference wavefield is defined, such that:

\[ u_0(x, \omega, x_s) = a(\omega)g_0(x, \omega, x_s) \, , \]  

(B4)

\[ L_0(x, \omega)g_0(x, \omega, x_s) = -\delta(x - x_s) \, . \]  

(B5)

By substituting Equation B2 into Equation B3, we find:

\[ L(x, \omega)u(x, \omega, x_s) = (L_0(x, \omega) + \Delta L(x, \omega))u(x, \omega, x_s) = -a(\omega)\delta(x - x_s) \, , \]  

(B6)

which is equivalent to:

\[ L_0(x, \omega)u(x, \omega, x_s) = -a(\omega)\delta(x - x_s) - \Delta L(x, \omega)u(x, \omega, x_s) \, . \]  

(B7)

Equation B7 can be solved by multiplying each side with \( g_0 \) and integrating. This produces:
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\[ u(x, \omega, x_s) = \int dx' g_0(x, \omega, x') [a(\omega) \delta(x' - x_s) + \Delta L(x', \omega)u(x', \omega, x_s)] . \quad (B8) \]

Equation B4 can be used to partially solve Equation B8, resulting in:

\[ u(x, \omega, x_s) = u_0(x, \omega, x_s) + \int dx' g_0(x, \omega, x') \Delta L(x', \omega)u(x', \omega, x_s) . \quad (B9) \]

Finally, by rearranging terms, we find the familiar version of the Lippmann-Schwinger equation:

\[ \Delta u(x, \omega, x_s) = \int dx' g_0(x, \omega, x') \Delta L(x', \omega)u(x', \omega, x_s) , \quad (B10) \]

where \( \Delta u(x, \omega, x_s) \) is described as shown in Equation B3.
APPENDIX C

Derivation of Born Reflectivity in Elastic Media

This Appendix closely follows the derivation presented in Shaw and Sen (2004) that outlines the theoretical framework for calculating a reflection coefficient \( R_{pp} \) using the Born approximation. Throughout this section Einstein index notation is used to imply summation over terms with the same indices.

The fundamental equation used in the Shaw and Sen (2004) workflow is the linearized Lippmann-Schwinger equation. The equation is used in a different form, but is equivalent to the 1st order Hybrid Born-Rytov Solution.

\[
\Delta u_n (x_r, \omega) = \int_V dx' \left( \omega^2 \Delta \rho (x') u_l^0 (x', \omega) G_n \left( x', \omega; x_r \right) - \Delta c_{ijkl} (x') \frac{\partial u_k^0 (x', \omega)}{\partial x_l} \frac{\partial c_{nl}^0 (x', \omega; x_r)}{\partial x_j} \right), \tag{C1}
\]

where \( \Delta u_n \) is the elastic wavefield perturbation (i.e., \( \Delta u = u - u^0 \)), \( u^0 \) is the reference wavefield, \( \omega \) is frequency, \( x \) is position, \( \Delta \rho \) is density perturbation (i.e., \( \Delta \rho = \rho - \rho^0 \)), \( \Delta c_{ijkl} \) is an elastic parameter perturbation (i.e., \( \Delta c_{ijkl} = c_{ijkl} - c_{ijkl}^0 \)), and \( G_n^0 \) is the ray theoretic Green’s function. A derivation of Equation C1 can also be found in Hudson (1981).

Initially, a Green’s function must be defined. The one shown below was used.

\[
G_n^P (x_r, \omega; x_s) = \frac{N_n N_i}{4\pi \rho_0 \alpha_0^2} \frac{1}{r} e^{i \omega r / \alpha_0},
\]

where \( \alpha_0 \) represents p-wave velocity and \( N_n \) and \( N_i \) represent source and receiver directions, respectively. Plugging in this Greens function into Equation C1 and rearranging terms produces

\[
\Delta u_{mn} (x_r, \omega; x_s) = N_m N_n \omega^2 \int_Z \left( \int_S \int_S' \int_S' \int_S' \left[ (\Delta \rho \delta_{ik} + \Delta c_{ijkl} p_j p_l) g_k g_l \right] A(r) e^{i \phi(r)} \right), \tag{C3}
\]

where \( \phi(r) \) is a phase term and

\[
A(r) = \frac{1}{\left(4\pi \rho_0 \alpha_0^2\right)^2} \frac{1}{r^2}. \tag{C4}
\]
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$A(r)$ represents the scaler amplitude at distance $r$. In Equation C3, the $p$ and $g$ variables represent polarization and propagation direction unit vectors for the incident and scattered wavefield (see Equation C19). The integral in Equation C3 can then be calculated using the method of stationary phase. This produces the following solution:

$$
\int \int d\xi_1 d\xi_2 \left[ (\Delta \rho \delta_{ij} + \Delta c_{ijkl} p_j p_i) g'_i g'_k \right] A(r) e^{i \omega \phi(r)} \bigg|_{r=r_0}.
$$

(C5)

The phase term in the above equations can be determined by taking the total distance traveled and dividing by the velocity. In the case of a scatterer in an otherwise uniform medium, the distance is the sum of the distance from the source to the scatterer and the distance from the scatterer to the receiver. Thus, the phase term has the following form:

$$
\varphi(r) = \frac{1}{\alpha_0} (|r'| + |r - r'|).
$$

(C6)

In order to estimate the stationary phase solution shown in Equation C5, we begin by analyzing a general Fourier transform of the type shown below in Equation C7.

$$
F(\omega) = \int_{-\infty}^{\infty} f(r) e^{i \omega \phi(r)} dr.
$$

(C7)

By expanding the phase term $\varphi$ around some stationary point using a Taylor expansion, we find

$$
\varphi(r) = \varphi(r_0) + \frac{1}{2} \nabla^2 \varphi(r) \bigg|_{r_0} (r - r_0)^2 + \cdots.
$$

(C8)

After dropping higher order phase terms, Equation C7 becomes:

$$
F(\omega) \approx f(r_0) e^{i \omega \varphi(r_0)} \int_{-\infty}^{\infty} e^{\frac{1}{2} \nabla^2 \varphi(r) \bigg|_{r_0} (r - r_0)^2} dr.
$$

(C9)

Equation C9 has a solution of the form (Shaw and Sen, 2004, Appendix A)
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\[ F(\omega) \approx \left( \frac{2\pi}{\omega} \right)^{\frac{m}{2}} \frac{m}{2} f(r_0) e^{i\omega\varphi(r_0) + i\text{sgn}(\omega)\pi/4} \frac{1}{\sqrt{\det A}}, \]  

(C10)

where \( A \) is the Hessian matrix. Hessian matrix \( A \) has the following form

\[ A_{ij} = \left( \frac{\partial^2 \varphi(r_0)}{\partial x_i \partial x_j} \right), \quad i, j = 1, 2, \ldots m. \]  

(C11)

By using the phase term of the form shown in Equation C6, the Hessian matrix determinant reduces to:

\[ \det A = \frac{4 \cos^2 i}{\omega_0^2 r_0^2} \]  

(C12)

Combining C12 and C10 produces a solution for C6.

\[ F(\omega) = \int_{-\infty}^{\infty} f(r) e^{i\omega\varphi(r)} dr \rightarrow F(\omega) = \frac{\frac{4\pi}{\omega_0^2} r_0^2}{\omega \cos(l)} f(r). \]  

(C13)

By substituting the general function term \( f(r) \) in Equation C13 with the terms under the double integral in Equation C3 produces the solution shown in Equation C5. Note that a similar procedure can be performed to calculate a semi-infinite Fourier integral using the stationary phase method:

\[ H(\omega) = \int_{\lambda}^{\infty} h(r) e^{i\omega\varphi(r)} d\tilde{r} \rightarrow H(\omega) = \left. \frac{1}{i\omega} \frac{h(r)}{\varphi'(r)} e^{i\omega\varphi(r)} \right|_{r=\lambda}. \]  

(C14)

By using the stationary phase solutions shown in Equations C12 and C14, Equation C3 is transformed to:

\[ \Delta u_{mn}(x_r, \omega; x_s) = -N_m M_n \frac{1}{4\pi \rho_0 \alpha_0^2 r_0} \left( \frac{1}{4\rho_0 \cos^2 i} S(r_0) \right) e^{i\omega\varphi(r_0)}, \]  

(C15)

where \( S(r_0) \) is the “scattering function” defined by:
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\[ S(r_0) = [\Delta \rho \delta_{ik} + \Delta \epsilon_{ijkl} p_j^l p_l] g_i^l g_k^l |_{r=r_0} \]  
\[(C16)\]

By expressing Equation C15 in terms of the Green’s function shown in Equation C2, we find

\[ \Delta u_{mn}(x_r, \omega; x_S) = -G_{mn}(x_r, \omega; x_S) R(i) \]  
\[(C17)\]

where \( R(i) \) describes seismic reflectivity and is defined by

\[ R(i) = \frac{1}{4 \rho_0 \cos^2 i} S(r_0) . \]  
\[(C18)\]

Thus, an approximation for reflectivity can be found in terms of the scattering potential shown in Equation C17 and C18. This requires knowledge of the polarization and propagation unit vectors. A simple geometric analysis in spherical coordinates indicates that:

\[ \theta_s = \theta_i \quad \text{and} \quad \phi_s = \phi_i + \pi , \]
\[ g_1 = \sin \theta_i \cos \phi_i \quad \text{and} \quad p_1 = g_1 / \alpha_0 , \]
\[ g_2 = \sin \theta_i \sin \phi_i \quad \text{and} \quad p_2 = g_2 / \alpha_0 , \]
\[ g_3 = \cos \theta_i \quad \text{and} \quad p_3 = g_3 / \alpha_0 , \]
\[ g'_1 = \sin \theta_s \cos \phi_s \rightarrow g'_1 = -\sin \theta_i \cos \phi_i \quad \text{and} \quad p'_1 = g'_1 / \alpha_0 , \]
\[ g'_2 = \sin \theta_s \sin \phi_s \rightarrow g'_2 = \sin \theta_i \sin \phi_i \quad \text{and} \quad p'_2 = g'_2 / \alpha_0 , \]
\[ g'_3 = \cos \theta_s \rightarrow g'_3 = \cos \theta_i \quad \text{and} \quad p'_3 = g'_3 / \alpha_0 . \]  
\[(C19)\]

The angles described in Equation C19 are visually described in Figure C1.
Combined, this information allows calculation of the first term of the Scattering function:

\[ [g_i g'_i]_{r=r_0} = g'_1 + g'_2 + g'_3 = \]
\[ - (\sin \theta \cos \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \theta)^2 = \cdots = \cos(2\theta) \quad \text{(C20)} \]

\[ S(r_0) = \Delta \rho [g_i g'_i]_{r=r_0} + \Delta c_{ijkl} [g'_j g_i g_k g_l]_{r=r_0} = \Delta \rho \cos(2\theta) + \Delta c_{ijkl} [g'_i g'_j g_k g_l]_{r=r_0}. \quad \text{(C21)} \]

Expanded, the 2nd term of the scattering function has the following form:

\[ \Delta c_{ijkl} [g'_j g'_i g_k g_l]_{r=r_0} = \Delta c_{1111} [g'_1 g'_1 g_1 g_1]_{r=r_0} + \]
\[ \Delta c_{1112} [g'_1 g'_1 g_1 g_2]_{r=r_0} + \cdots + \Delta c_{3333} [g'_3 g'_3 g_3 g_3]_{r=r_0}. \quad \text{(C22)} \]

The second term of the scattering function can be calculated by assuming that the \( \Delta c_{ijkl} \) term has form shown in Equation C23. Note that in C23 the \( \Delta c_{ijkl} \) term has been replaced by \( \Delta c_{mn} \), where \( n \) and \( m \) are the equivalent Voight notation values for \( ij \) and \( kl \), respectively.

\[ \Delta c_{ijkl} \rightarrow \Delta c_{mn} = \begin{bmatrix} \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & 0 & 0 & 0 \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & \Delta c_{33} - 2\Delta c_{55} & 0 & 0 & 0 \\ \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} - 2\Delta c_{55} & \Delta c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta c_{55} \end{bmatrix}. \quad \text{(C23)} \]
NOTE: the tensor shown in Equation C23 is valid for isotropic media. However, by assuming an anisotropic stiffness tensor and an anisotropic $\Delta c_{mn}$ this same procedure could be used to develop an approximation for $R_{PP}$ that is valid in anisotropic media. We plan on using this methodology when deriving anisotropic solutions in our future work.

When calculated, the second term of the scattering function reduces to

$$\Delta c_{ijkl} \left[ g_i^j p_k^l \right]_{r=r_0} = \frac{1}{\alpha_0} \left( \Delta c_{33} - 2 \Delta c_{55} \sin^2(2\theta_i) \right) . \quad (C24)$$

Note that Equation C24 has a factor of 2 in front of the $\Delta c_{55}$ term that is not present in Shaw and Sen’s paper. This is due to a typo in their original paper. Combining the 1$^{st}$ and 2$^{nd}$ term solutions for the scattering function, we get

$$S(r_0) = \Delta \rho [g_i g_i']_{r=r_0} + \Delta c_{mn} \left[ g_i^j p_k^l \right]_{r=r_0} = \rho \cos(2\theta_i) + \frac{1}{\alpha_0} \left( \Delta c_{33} - 2 \Delta c_{55} \sin^2(2\theta_i) \right). \quad (C25)$$

Thus, combining Equation C25 and C18 provides the following estimate for reflectivity as a function of incident angle in an isotropic medium:

$$R_{PP}(\theta_i) = \frac{\Delta \rho \cos(2\theta_i) + \frac{1}{\alpha_0} \left( \Delta c_{33} - 2 \Delta c_{55} \sin^2(2\theta_i) \right)}{4\rho_0 \cos^2 \theta_i} . \quad (C26)$$